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2. A summary of approximately 50 words must accompany the paper, a 300-word synopsis must precede it, and a set of conclusions must end it.

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7. Illustrations must be drawn in black ink on one side of 8½-inch by 11-inch paper. Because illustrations will be reproduced with a width of between 3-inches and 4½-inches, the lettering must be large enough to be legible at this width. Photographs should be submitted as glossy prints. Explanations and descriptions must be made within the text for each illustration.

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SAFETY, RELIABILITY AND STRUCTURAL DESIGN[a]

By A. M. Freudenthal,[1] F. ASCE

SYNOPSIS

In the safety analysis of any structure the expected operational life for which it is, or should be designed, represents an important design parameter, the significance of which is not sufficiently understood. The concept of design for infinite life is as unrealistic as that of design for zero probability of failure, since the design life and the probability of failure are related through the "reliability function" of the designed structure. By expressing the probability of survival as a function of operational life, this function permits a rational approach to the safety analysis of modern structures for the criteria of ultimate load failure and of fatigue, as well as for the combination of both criteria.

In reality the safety factor is shown to be a statistical distribution function of the quotient of resistance and load considered as statistical variables. No engineering significance can, therefore, be attached to any specific single value of this factor other than as a "measure of location" (mean, median, made) of the statistical distribution. However, the probability or risk of failure or of survival is a unique parameter of the distribution function of the safety factor, and should replace the conventional concept of safety or rather be considered as its only rational quantitative measure. Its generalization in the form of the "reliability function" provides the basis for an integrated safety analysis of

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[a] This research was supported in part by the United States Air Force under Contract AF-33(616)-6288, monitored by the Materials Laboratory Wright Air Development Division.

[1] Prof. Civ. Engrg., Columbia Univ., New York, N. Y.

structures permitting quantitative differentiation between the safety of non-redundant and redundant structures.

INTRODUCTION

The increasing severity of operating conditions of modern engineering structures and the increasing complexity of their structural action makes it increasingly difficult to design such structures for assumedly infinite, but in fact only unspecified, operational life without adversely affecting either their performance, their cost, or both. Moreover, the growing use of materials of high strength but unimproved fatigue performance increases the fatigue sensitivity of such structures and requires the development of methods of safety analysis in which the criteria of ultimate-load failure and failure under repeated load (fatigue proper or alternating plasticity) are considered jointly. Such methods must obviously be based on a quantitative measure of safety that retains its validity under both failure conditions. With the aid of such a measure of safety, joint effect of the combination of failure criteria can be adequately expressed.

This measure is the probability of survival ("reliability") L or of failure ("mortality") P of the structure at the termination of the operational life for which it has been designed or, if the operational life has not been specified, as a function of the (variable) operational life. For conditions of ultimate load failure as well as of fatigue failure, the operational life is expressed in terms of the expected number N of load repetitions throughout the service life of the structure. Thus the "reliability function" $L(N)$ or its complement, the "mortality function" $P(N) = 1 - L(N)$, expressing, respectively, the probabilities of survival or of failure, as functions of the momentary "age" N of the structure, provide effective quantitative measures of safety that are valid for either failure criterion as well as for a combination of both of them.

Such a combination is most expediently based on the concept of the "risk" of failure or the "failure rate" $r(N)$, which expresses the probability of failure at the $(N + 1)^{th}$ load repetition of those structures or structural parts of the initial statistical "population" considered that have survived N load repetitions, and have thus reached the "age" N. According to this definition, the "risk function" $r(N)$ is the probability of failure in the statistical population at the N^{th} load cycle $p(N)$, referred to the surviving percentage of the population at the N^{th} cycle $L(N)$. Because the relation between the "probability density" $p(N)$ and the "cumulative probability" or reliability function $L(N)$ is

$$p(N) = -\frac{dL(N)}{dN} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

the "risk function" can be stated as

$$r(N) = \frac{p(N)}{L(N)} = -\frac{1}{L(N)}\,\frac{dL(N)}{dN} = -\frac{d}{dN}\log L(N) \quad \ldots\ldots (2)$$

Therefore, by integration

$$L(N) = \exp\left[-\int_0^N r(z)\,dz\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

In statistical theory the risk function is known as the intensity function, and its inverse as the Mills ratio.[2a]

While the reliability function L(N) is a purely statistical measure of dispersion, the risk function r(N) is open to physical (or engineering) interpretation. Under conditions of ultimate load failure the "risk" of failure is, by definition, independent of the "age" of the structure and therefore constant, while the nature of fatigue failure implies a "risk" or "rate" of failure that increases with "age." To total "risk" under the combination of the two failure criteria is simply the sum of the "risks" associated with the individual criteria. Denoting by P_F' the (constant) probability or risk of ultimate load failure under a single load application and by $P_F''(N)$, the (increasing) risk of

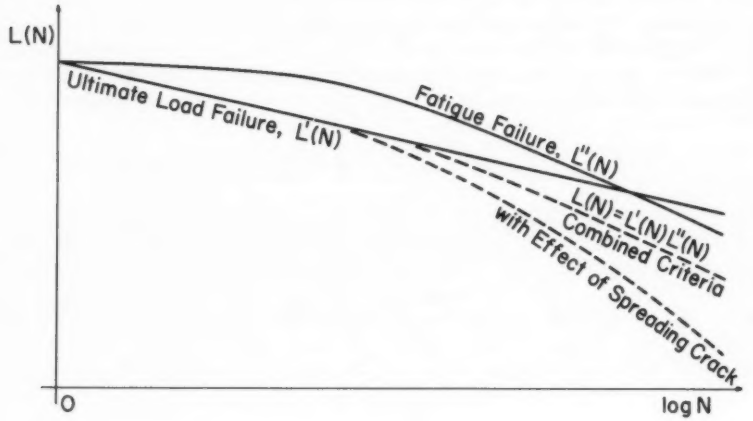

FIG. 1.—RELIABILITY FUNCTIONS L (N) FOR ULTIMATE LOAD FAILURE, FATIGUE, AND A COMBINATION OF BOTH

fatigue failure at the N-th load application, the combined risk of failure[3] becomes

$$P_F(N) = P_F' + P_F''(N) \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and, therefore, the associated reliability function is

$$L(N) = \exp(-NP_F')\, L''(N) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

for which the form of the reliability function for fatigue failure

$$L''(N) = \exp\left[-\int_0^N P_F''(z)\, dz\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

depends on the form of the risk function $P_F''(N)$ and vice versa (Fig. 1).

---

2 "Statistics of Extremes," by E. J. Gumbel, Columbia Univ. Press, New York, N. Y., 1958, (a) p. 20; (b) pp. 75–155; (c) pp. 272–302; (d) pp. 255–272.

3 "Prediction of Fatigue Failure," by A. M. Freudenthal, Journal of Applied Physics, Vol. 31, December, 1960.

Eqs. 4 and 5 are based on the simplifying assumption that P_F' and $P_F''(N)$ are independent, which implies that the possibility of gradual reduction of the ultimate carrying capacity by spreading fatigue damage as well as that of fatigue damage produced by load-intensities within the ultimate failure-load range are both neglected. Without this assumption, P_F' must be introduced as a function $P_F'(N)$ of N, considering the effect of fatigue-crack propagation on the "static" carrying capacity. In addition $L''(N)$ would have to be determined from fatigue tests under variable load amplitudes including not only the range of operational loads up to the "limit-load" of serviceability (that might occur at most once or a few times during the service life of the structure), but also the extremely rare load-amplitudes exceeding this limit, but without causing failure on a single application. Such fatigue tests are not only impractical because of their duration, but the effects of load amplitudes that are most unlikely to occur during the expected life are, in fact, negligibly small. However, the effect of spreading fatigue damage on P_F' can only be neglected as long as $P_F'' \ll P_F'$. When P_F'' and P_F' are of comparable order of magnitude, a function $P_F'(N)$ rather than a constant value P_F' must be considered.

SAFETY ANALYSIS FOR ULTIMATE LOAD CRITERION

The probability of failure $P_F' = P_F$ under a single load application refers to a statistical "population" of nominally identical structures of (statistical) carrying capacity R, each of which has been subjected to one application of a (statistical) load S. This represents the ratio of structures that are expected to fail, or the probability of any one of those structures to fail, under the single load application. If the "safety-factor" of a structure within the considered "population" is defined as the ratio ν of its resistance R to the applied load S;

$$\nu = \frac{R}{S} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7a)$$

This ratio is different for each structure of the "population." It is, in fact, a statistical variable with probability density $p(\nu)$ and cumulative function $P(\nu) = \int_0^{\nu} p(\nu)\, d\nu$, which is completely determined by the probability densities of the carrying capacity $p_1(R)$, and of the load population $p_2(S)$.

The term R denotes the carrying capacity of the structure or part of the structure as determined by the relevant material properties and the geometric characteristics associated with the failure criterion, such as areas, section moduli, static moments and spans, as well as the effect of connections (stress-concentration factors). It has the same dimension as the load S, which represents the entity of possible load intensities that might be encountered by the structures, even if their occurrence is quite unlikely within the period of service for which the structure is designed. Whereas R is always a statistical variable, there are types of structures for which S may be represented by a non-statistical maximum load S_{max}. This is because it is a true service load of high frequency of occurrence, such as loads in ware-houses, in bulk-and fluid-containers, on cranes, railroad-bridges, and so on. For such structures the distribution $p(\nu)$ of

$$\nu = \frac{R}{S_{max}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7b)$$

is identical with the distribution $p_1(R)$ divided by the constant S_{max}.

Because failure occurs when $S > R$ or $\nu < 1$, the probability of failure P_F is identical with the probability that $\nu < 1$ or $P_F = P(1)$, which is the ordinate of the cumulative function $P(\nu)$ at $\nu = 1$. The relationship

$$P_F = \int_0^1 p(\nu)\, d\nu = \int_0^1 p\left(\frac{R}{S}\right) d\left(\frac{R}{S}\right) = P(1) \quad \ldots\ldots\ldots (8)$$

determines the probability of failure directly in terms of the probability densities $p_1(R)$ and $p_2(S)$ without reference to a specific value of the "safety factor."

The definition of the safety factor ν in terms of a statistical distribution function $p(\nu)\, d\nu$ implies that any value $0 < \nu < \infty$ is possible. It reduces the safety analysis of a structure or structural part to the determination, on the basis of Eq. 7a, of the function $p(\nu)$ and the integration specified in Eq. 8. If the distribution $p(\nu)$ is unimodal, values within a certain "central" range (around the mode) are the most likely, whereas the likelihood of values outside this range decrease with increases in their distance from the central range. Specific values of ν acquire a meaning only as they represent a "measure of location" of the central range, such as the mean $\overline{\nu}$, mode $\tilde{\nu}$, or median $\breve{\nu}$ of the distribution $p(\nu)$. A conveniently selected "measure of location" can be considered as a nominal or "central" safety factor ν_0 representing the ratio of the central values (R_0/S_0). The selection of this ratio will depend on the character of the functions $p_1(R)$ and $p_2(S)$. Thus, if these functions are normal, their "central" values are the means $\overline{R}$ and $\overline{S}$. If the values are logarithmic-normal, the central values, being the means of $(\log R)$ and $(\log S)$, are the medians $\breve{R}$ and $\breve{S}$ at the probability level $P = 0.5$, since $\overline{\log R} = \log \breve{R}$ and $\overline{\log S} = \log \breve{S}$. If the functions are "extremal,"[2b] their central values are the "characteristic" values $\tilde{R}$ and $\tilde{S}$, equal to or close to the modes at the probability levels $P = (1 - 1/e)$ and $(1/e)$, respectively.

Two types of structures must be considered:

1. Structures in which S is independent of R, so that the ratio $\nu = R/S$ is one between independent statistical variables. It can, therefore, be obtained by relatively simple statistical procedures.

2. Structures in which S depends on R through the effect of R on the flexibility and inertia. The ratio ν is, therefore, one between functionally dependent variables and cannot be derived unless the functional dependence is considered in the statistical analysis.

Structures of the first type are those for which stress-analysis can be based on the undeformed shape ("rigid" structures). The intensity of the forces for structures of the second type depends significantly on the deformational response of the structure which, in turn, is affected by its carrying capacity R ("flexible," "aero-elastic," and "hydro-elastic" structures). Only structures of the first type will be considered.

The principal feature of a rational approach to the safety analysis of such structures is the consideration of the safety factor as a statistical variable of probability density $p(\nu)$ determined by the probability densities of carrying capacity $p_1(R)$ and of load $p_2(S)$. If $p_1(R_S)$ refers to a criterion of "unserviceability,"[4] the (statistical) safety factor $\nu = \nu_S$ and probability $P_F = P_{FS}$ refer to the same criterion. If $p_1(R_F)$ refers to a criterion of actual failure (fracture, plastic collapse) the associated (statistical) safety factor $\nu = \nu_F$

4 "Synopsis of First Progress Report, of the Committee on Factors of Safety," by O. G. Julian, Proceedings, ASCE, Vol. 83, No. ST 4, July, 1957.

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[4] "Synopsis of First Progress Report, of the Committee on Factors of Safety," by O. G. Julian, Proceedings, ASCE, Vol. 83, No. ST 4, July, 1957.

and probability P_F are those of such failure. Obviously, the probability density of the load $p_2(S)$ is usually the same for both criteria. Since the central locations $R_{S0} < R_{F0}$, the central safety factors $\nu_{S0} < \nu_{F0}$ and, therefore, $P_{FS} > P_F$.

Considering two independent, non-negative, continuous statistical variables R and S with probability densities $p_1(R)$ and $p_2(S)$ and (cumulative) probability functions $p_1(R)$ and $p_2(S)$, the joint distribution of the two variables $p(R,S)$ is the product

$$p(R,S)\ dR\ dS = p_1(R)\ p_2(S)\ \ dR\ dS \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

Introducing the quotient $\nu = (R/S)$ yields

$$R = \nu\ S \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10a)$$

and

$$\frac{dR}{d\nu} = S \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (10b)$$

The distribution

$$p_1(\nu S)\ p_2(S) \left(\frac{dR}{d\nu}\right) d\nu\ dS = p_1(\nu S)\ p_2(S)\ S\ d\nu\ dS \quad \ldots\ldots\ldots \quad (11)$$

is the joint distribution $p(\nu,S)$ of the quotient ν and the variable S. By integrating over S between $0 \leq S \leq \infty$, the probability density $p(\nu)$ is obtained as the "marginal" distribution of $p(\nu,S)$:[5]

$$p(\nu) = \int_0^\infty p_1(\nu S)\ p_2(S)\ S\ dS \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

The probability function is

$$P(\nu) = \int_0^\nu p(\nu)\ d\nu = \int_0^\nu \int_0^\infty S\ p_1(\nu S)\ \ p_2(S)\ dS\ d\nu \quad \ldots\ldots \quad (13)$$

Changing the order of integration yields

$$P(\nu) = \int_0^\infty \int_0^{\nu S} p_2(S)\ p_1(\nu S)\ \ dS =$$

$$\int_0^\infty P_1(\nu S)\ p_2(S)\ dS = \int_0^1 P_1(\nu S)\ dP_2(S) \quad \ldots\ldots\ldots \quad (14)$$

Eq. 14 represents the relation between $P(\nu)$ and the known probability functions of R and S. The probability of failure is, therefore,

$$P_F = P(\nu)_{\nu=1} = \int_0^1 P_1(S)\ dP_2(S) \quad \ldots\ldots\ldots\ldots \quad (15)$$

Eq. 14 can be evaluated either directly or by numerical integration. It should be noted that this equation is closely related to the Mellin transforms[6] of the functions $p_1(R)$ and $p_2(S)$;

$$g_1(s) = \int_0^\infty p_1(R) R^{s-1}\ dR \quad \ldots\ldots\ldots\ldots \quad (16a)$$

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5 "Methods of Safety Analysis of Highway Bridges," Sixth Congress Internatl. Assn. of Bridge and Structural Engrs., Stockholm, 1960, p. 655.

6 "Some Applications of the Mellin Transform in Statistics," by B. Epstein, Annual of Mathematical Statistics, Vol. 19, 1948, p. 370.

and

$$g_2(s) = \int_0^\infty p_2(S) S^{s-1}\, dS \quad \ldots\ldots\ldots\ldots\ldots \quad (16b)$$

which represent the $(s-1)^{th}$ moments of the distributions $p_1(R)$ and $p_2(S)$ or, in statistical terminology, the expected values or "expectations" $E\left(R^{s-1}\right)$ and $E\left(S^{s-1}\right)$ of R^{s-1} and S^{s-1}, respectively. Because R and S are independent variables, the "expectation" $E\left(\nu^{s-1}\right)$ of $\nu^{s-1} = (R/S)^{s-1}$ considered as a product $\nu^{s-1} = R^{s-1}\,(1/S)^{s-1}$, is equal to the product of the expectations $E\left(R^{s-1}\right)$ and $E\,(1/S)^{s-1} = E\left(S^{1-s}\right)$. Hence, the Mellin transform

$$g(s) = \int_0^\infty p(\nu)\nu^{s-1}\, d\nu \quad \ldots\ldots\ldots\ldots\ldots \quad (17)$$

of the distribution $p(\nu)$ is the expectation

$$E\left(\nu^{s-1}\right) = E\left(R^{s-1}\right)\, E\left(S^{1-s}\right) = g_1(s)\; g_2(2-s) = g(s) \ldots\ldots\ldots \quad (18)$$

in which $E\left(S^{1-s}\right) = E\left(1/S^{s-1}\right) = g_2(2-s)$ is the reciprocal moment of order $(s-1)$ of the function $p_2(S)$. Inversion of Eq. 18 produces the distribution $p(\nu)$ directly.

The Mellin transform $G(s)$ of the probability function $P(\nu)$ is obtained with the aid of Eq. 14;

$$G(s) = \int_0^\infty \nu^{s-1} \int_0^\infty P_1(\nu S)\, p_2(S)\, dS d\nu = G_1(s)\; g_2(1-s) \ldots\ldots\ldots (19)$$

in which $G_1(s)$ is the Mellin transform of the probability function $p_1(R)$, and $g_2(s)$ that of the probability distribution $p_2(S)$. Inversion of $G(s)$ produces the probability function $P(\nu)$.

A particularly simple illustration of the procedure of deriving $p(\nu)$ from $p_1(R)$ and $p_2(S)$, without the use of Eqs. 12 and 14, arises when $p_1(R)$ and $p_2(S)$ are both logarithmic-normal with median values of R and S and standard deviations δ_R and δ_S of log R and log S, respectively. Since $\nu = R/S$, or

$$\log \nu = \log R - \log S \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

it follows that log ν, as the difference of two normal distributions, is also normally distributed, with mean

$$\log\left(\frac{\breve{R}}{\breve{S}}\right) = \log \breve{\nu} = \log \nu_0 \ldots\ldots\ldots\ldots\ldots \quad (21)$$

and standard deviation

$$\delta = \sqrt{\delta_R^2 + \delta_S^2} \ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

Therefore

$$P(\nu) = \Phi\left[\frac{\log\left(\nu/\nu_0\right)}{\delta}\right] \ldots\ldots\ldots\ldots\ldots \quad (23)$$

in which

$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt \ldots\ldots\ldots\ldots\ldots \quad (24)$$

The probability of failure is thus obtained directly from tables of error functions at

$$P_F = \Phi\left[-\frac{\log \nu_0}{\delta}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

Eq. 25 shows the dependence of P_F on ν_0 and δ. A reduction of P_F requires an increase of ν_0, a reduction of δ or both. An increase of ν_0 alone remains ineffective if at the same time δ increases. This is frequently the case when structural materials of higher strength that show an increased scatter of the strength properties are used.

In Fig. 2, Eq. 25 is presented for various combinations of the ratios $\sigma_S/\check{S}$ and $\sigma_R/\check{R}$ in which σ_S and σ_R denote the standard deviations of S and R, re-

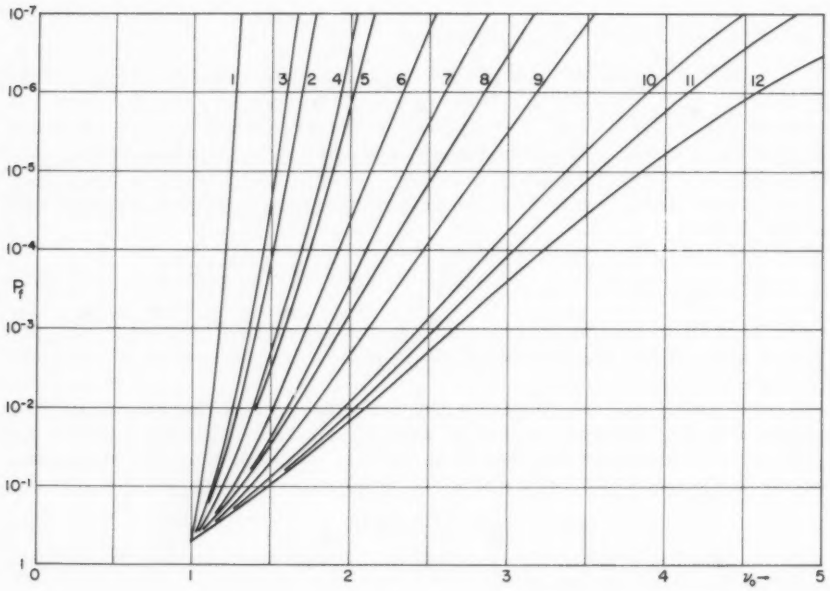

FIG. 2.—PROBABILITY OF FAILURE AS FUNCTION OF CENTRAL SAFETY FACTOR (LOGARITHMIC-NORMAL DISTRIBUTIONS)

spectively. Fig. 3 presents the results of the numerical evaluation of Eq. 15 under the assumption that $P_1(R)$ is the extremal distribution of smallest values[2c] ("Weibull distribution")

$$P_1(R) = 1 - \exp\left[-(R/\check{R})^{\alpha} R\right] \quad \ldots\ldots\ldots\ldots \quad (26a)$$

whereas $P_2(S)$ is the extremal distribution of largest values[2d] ("Frechet distribution")

$$P_2(S) = \exp\left[-(S/\check{S})^{-\alpha} S\right] \quad \ldots\ldots\ldots\ldots \quad (26b)$$

for various combination of the ratios $\sigma_S/\check{S}$ and $\sigma_R/\check{R}$, in which $\tilde{S}$ and $\tilde{R}$ are the "characteristic values" and α_S and α_R are scale parameters that are inverse functions of δ_S and δ_R. Table 1 presents the values for these curves.

Introducing Eqs. 26 and 27 into Eq. 14 the following expression is obtained:

$$P(\nu) = 1 - \int_0^\infty \exp\left[-t - \left(\frac{\nu}{\nu_0}\right)^\alpha t^{-\frac{\alpha}{\beta}}\right] dt \quad \ldots\ldots\ldots\ldots \quad (27a)$$

in which $t = (S/\tilde{S})^{-\beta}$.

Therefore, the probability of failure is

$$P_F = P(1) = 1 - \int_0^\infty \exp\left[-t - \left(\frac{1}{\nu_0}\right)^\alpha t^{-\frac{\alpha}{\beta}}\right] dt \quad \ldots\ldots\ldots \quad (27b)$$

This integral can only be evaluated by numerical integration. The results have been presented in Fig. 3 for various ratios $\sigma_S/\tilde{S}$ and $\sigma_R/\tilde{R}$.

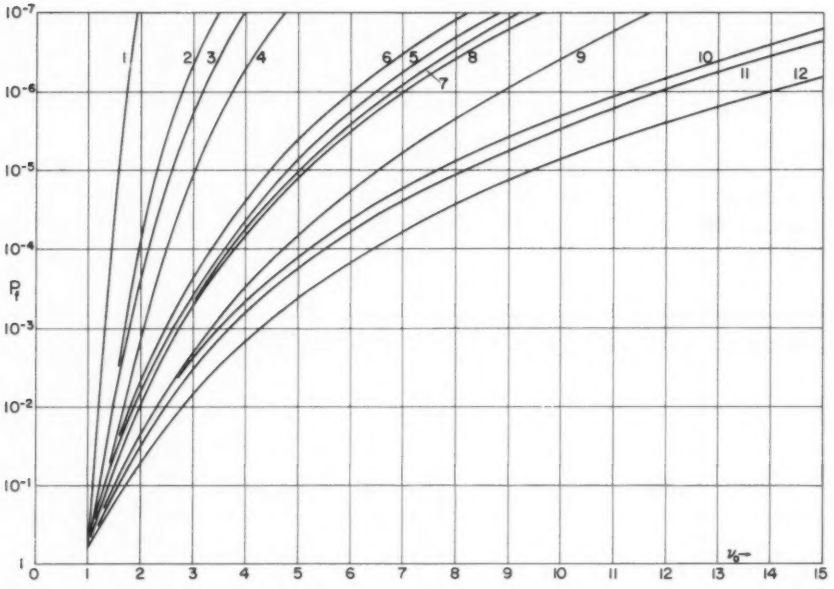

FIG. 3.—PROBABILITY OF FAILURE AS FUNCTION OF CENTRAL SAFETY FACTOR (EXTREME-VALUE DISTRIBUTIONS)

For a non-statistical load $S = S_{max}$ and $\sigma_S = 0$. Eq. 27b can be expressed in closed form

$$P_F = 1 - \exp\left[-\nu_0^{-\alpha}\right] \quad \ldots\ldots\ldots\ldots\ldots \quad (27c)$$

The parameters δ_S or δ_R and α_S, α_R are related to the coefficients of variation $\sigma_S/\check{S}$ or $\sigma_R/\check{R}$ and $\sigma_S/\tilde{S}$, $\sigma_R/\tilde{R}$ in Table 2. These relationships permit the selection of parameters to represent observations of σ_S and σ_R.

Whether logarithmic-normal or extremal distributions are to be applied in a particular safety analysis will depend on the level of the control imposed on the production of the structural material and on the construction process, as well as the extent of variability of the load.[7]

Logarithmic-normal distributions have been found to fairly well represent material parameters obtained under the conditions of stringent plant control characteristic for manufacturing process. Extremal distributions tend to arise when the level of quality control is only fair to poor, as on construction sites. With respect to loads, logarithmic-normal distributions are likely to represent

TABLE 1.—PROBABILITY OF FAILURE AS FUNCTION OF CENTRAL SAFETY FACTOR

Curve (1)	$\sigma_S/\breve{S}$ (2)	$\sigma_R/\breve{R}$ (3)
(a) Fig. 2: logarithmic normal distributions		
1	0.00	0.05
2	0.10	0.05
3	0.00	0.10
4	0.10	0.10
5	0.00	0.15
6	0.10	0.15
7	0.20	0.05
8	0.20	0.10
9	0.20	0.15
10	0.30	0.05
11	0.30	0.10
12	0.30	0.15
(b) Fig. 3: extreme-value distributions		
1	0.00	0.05
2	0.10	0.05
3	0.00	0.10
4	0.10	0.10
5	0.00	0.15
6	0.20	0.05
7	0.20	0.10
8	0.10	0.15
9	0.20	0.15
10	0.30	0.05
11	0.30	0.10
12	0.30	0.15

the complete load population, while extremal distributions are obtained if a new load population is derived therefrom, containing only loan intensities exceeding a relatively high threshold-value. This new population will have a higher cen-

[7] "Safety and the Probability of Structural Failure," by A. M. Freudenthal, Transactions, ASCE, Vol. 121, 1956, p. 1337.

tral location $\tilde{S}$ than that of the parent population $\check{S}$, while its standard deviation will be appreciably lower.

ULTIMATE LOAD CRITERION OF REDUNDANT STRUCTURES

When the considered structure has a single load-path and consists of n parts of "resistance" R, each of which is subject to the full load S (Fig. 4), failure of the structure will occur when the weakest of the parts fails (non-redundant chain-structure). Denoting by p_{FK} the probability of failure of the individual member, the probability of survival of all n members is

$$L(n) = \prod_{k=1}^{k=n} (1 - p_{Fk}) \sim 1 \sum_{k=1}^{k=n} p_{Fk} \quad \ldots\ldots\ldots\ldots \quad (28)$$

therefore, the probability of failure of the structure is

$$P_F = 1 - \prod_{k=1}^{k=n} (1 - p_{Fk}) \sim \sum_{k=1}^{k=n} p_{Fk} \quad \ldots\ldots\ldots\ldots \quad (29)$$

The survivorship function for N load repetitions is

$$L(N) = \prod_{1}^{n} \left[\exp\left(-Np_{Fk}\right)\right] = \exp\left(-N\, P_F\right) \quad \ldots\ldots\ldots\ldots \quad (30)$$

which is identical with that for the single-member structure, provided the individual probabilities p_{Fk} fulfill the condition of Eq. 29. When all members

TABLE 2

logarithmic-normal				extremal			
$\sigma_S/\check{S}$ or $\sigma_R/\check{R}$ =	0.05	0.10	0.15	$\sigma_S/\tilde{S}$ =	0.10	0.20	0.30
δ_S or δ_R =	0.022	0.043	0.064	α_S =	14.3	7.8	5.8
σ_S/S or $\sigma_R/\check{R}$ =	0.20	0.30		$\sigma_R/\tilde{R}$ =	0.05	0.10	0.15
δ_S or δ_R =	0.084	0.123		α_R =	24.4	11.8	7.4

are of equal probability of failure p_{Fk}, this condition has the form $p_{Fk} = 1/n\, P_F$. The individual members of an n-member chain-structure must, therefore, be designed with a higher "central" safety factor ν_0 than an equivalent single-member structure, in order to achieve the lower probability of failure $p_{Fk} < P_F$.

A statically determinate structure of n members is equivalent in structural action to a chain-structure, as the failure of any one of its n members produces failure of the structure as a whole. Thus, the probability of survival is expressed by Eq. 28, the probability of failure by Eq. 29, and the reliability function by Eq. 30. The only difference is that different members are subject to different stress-intensity spectra arising from the same external load-spectrum, so that the variation in the values p_{Fk} for individual members are likely to be much wider than in a simple chain-structure of nominally identical, iden-

tically loaded members. Thus, the probability of failure of the members in which the stresses remain low, even when the stresses in the critical members are in the failure range, may be so small as to be negligible in the sum of Eq. 29. The probability of failure P_F is therefore composed of the probabilities p_{Fk} of the critical members only rather than of all members.

When the considered structure is a multiple-load-path structure of n members and m load path (Fig. 5), in which (m/n) is an integer, the probability of

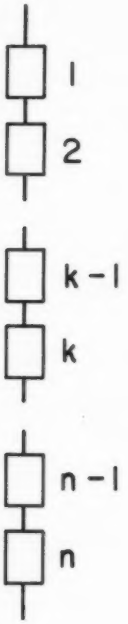

FIG. 4.—MODEL OF NON-REDUNDANT (SINGLE-LOAD-PATH) STRUCTURE

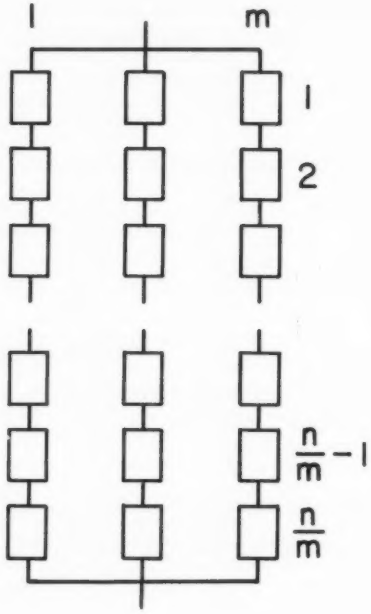

FIG. 5.—MODEL OF REDUNDANT (MULTIPLE-LOAD-PATH) STRUCTURE

failure of each of the m component chain structures subject to $(1/m)^{th}$ of the total load is initially given by the expression

$$P_{FO} = 1 - L_0\left(\frac{n}{m}\right) = 1 - \prod_{k=1}^{k=\frac{n}{m}} \left(1 - p_{F0k}\right) \sim \sum_{k=1}^{k=\frac{m}{n}} p_{F0k} \quad \ldots\ldots\ldots \quad (31)$$

Failure of the structure itself implies consecutive failure of the m component chain-structures in the course of which the probabilities p_{Fk} in members of the unbroken chains are suddenly increased as one after another of the chains fails. This increase is due to the increase of the ratio of the total load S to be carried by the unbroken chains. The increase in S reduces the "central" safety factor of the chain members from their initial value $\nu_{0k} = R_{0k}\, m/S_0$ to

$R_{0k}(m-1)/S_0, R_{0k}(m-2)/S_0, \ldots, R_{0k}/S_0$. Assume that the carrying capacity of all n members belongs to the same statistical population R so that $\nu_{0k} = \nu_0$ and p_{Fk} is constant. The increase of p_{Fk} from the initial values p_{F0k} for the n members of the m unbroken chains to $p_{F1k} < p_{F0k}$ for the $n(1 - 1/m)$ members, after the first chain is broken, and subsequently to $p_{F2k} < p_{F1k}$, and so on can be obtained from the relevant diagrams $P_F(\nu_0)$. Logarithmic normal distributions of R and S are obtained from Eq. 25, introducing the step-wise decreasing values of ν_0 associated with each chain failure.

As p_{F0k} increases to $p_{F1k}, p_{F2k}, \ldots p_{F(m-1)k}$ the probability of failure of each of the unbroken chains increases from P_{F0} according to Eq. 31 to

$$\left.\begin{aligned} P_{F,1} &= 1 - L_1\left(\frac{n}{m}\right) = 1 - \prod_{k=1}^{k=\frac{n}{m}} (1 - p_{F1k}) \sim \sum_{k=1}^{k=\frac{n}{m}} p_{F1k} \\ P_{F,2} &= 1 - L_2\left(\frac{n}{m}\right) = 1 - \prod_{k=1}^{k=\frac{n}{m}} (1 - p_{F2k}) \sim \sum_{k=1}^{k=\frac{n}{m}} p_{F2k} \\ P_{F,(m-1)} &= 1 - L_{m-1}\left(\frac{n}{m}\right) \sim \sum_{k=1}^{k=\frac{n}{m}} p_{F(m-1)k} \end{aligned}\right\} \quad \ldots\ldots (32)$$

The probability of failure of the structure is the probability of consecutive failure of all component chains

$$P_F = P_{F,0}\ P_{F,1} \cdot \cdots\ P_{F,(m-1)} \quad \ldots\ldots\ldots\ldots\ldots (33)$$

The probability of survival of the m-load-path structure is

$$L(n) = 1 - P_F = 1 - P_{F,0} \cdot P_{F,1} \cdots P_{F,(m-1)} \quad \ldots\ldots\ldots (34)$$

as compared to Eq. 28 for the single-load-path structure of the same total number of members.

Comparison of Eq. 33 with the probability $P_F = P_{F,0}$ of failure of a single load-path structure of (n/m) members suggests that the addition of alternative load-path reduces the probability of failure under a single load application significantly provided at least $P_{F,1} \ll 1$. If the failure of the first load-path (chain) already brings the probability $p_{F,1}$ close to unity, little is gained by the addition of an alternative load path.

The reliability function is obtained by a similar combination of the reliability functions of the individual chains at the different stages of chain-failure:

$$\left.\begin{aligned} L_0(N) &= \prod_{1}^{n/m} \left[\exp\left(-N\, p_{F0k}\right)\right] = \exp\left(-N\, P_{F,0}\right) \\ L_1(N) &= \exp\left(-N\, P_{F,1}\right) \\ L_{(m-1)}(N) &= \exp\left(-N\, P_{F,(m-1)}\right) \end{aligned}\right\} \quad \ldots\ldots (35)$$

The mortality function of the structure is obtained from the mortality functions of the component chains in the form of the product

$$P(N) = \left[1 - L_0(N)\right]\left[1 - L_1(N)\right] \ldots \left[1 - L_{(m-1)}(N)\right] \ldots\ldots (36)$$

so that the reliability function of the entire structure is

$$L_0(N) = 1-P(N) = 1-\left[1-\exp\left(-NP_{F,0}\right)\right] \ldots \left[1-\exp\left(-NP_{F,m-1}\right)\right] \quad (37)$$

This compares with Eq. 30 with $P_F = P_{F,0}$ for the single load path structure. For instance, for a three-load-path structure

$$L_0(N) = e^{-NP_{F,0}} + \left(1-e^{-NP_{F,0}}\right)e^{-NP_{F,1}} + \left(1-e^{-NP_{F,0}}\right)\left(1-e^{-NP_{F,1}}\right)e^{-NP_{F,2}} \quad (38)$$

Assume logarithmic-normal distributions of carrying capacity and of load with $\sigma_S/\check{S} = 0.30$, $\sigma_R/\check{R} = 0.10$, and $P_{F,0} = 4 \times 10^{-7}$ for $\nu_0 = 4.5$ according to curve 11 in Fig. 2. By reducing ν_0 from $\nu_0 = 4.5$ to 3.0 failure of one chain reduces $P_{F,0}$ to $P_{F,1} = 10^{-4}$, while failure of the second chain further reduces the central safety factor to $\nu_0 = 1.5$ and the probability of chain failure to $P_{F,2} = 10^{-1}$. With these values Eq. 38 can be written in the form

$$L_0(N) = e^{-4\times10^{-7}N} + \left(1-e^{-4\times10^{-7}N}\right)e^{-10^{-4}N} + \left(1-e^{-4\times10^{-7}N}\right)\left(1-e^{-10^{-4}N}\right)e^{-10^{-1}N}$$

The increase in the long-range reliability of the structure as compared to that of the single chain given by the first term is relatively small. For the second term it only extends over the range of values of $10^1 < N < 10^5$, whereas for the third term it is for $1 < N < 100$. Beyond these ranges the last two terms do not contribute.

The principal effect, however, is that the three-load-path structure assures that the failure of one-load-path still leaves a structure with a sufficiently high probability of short range survival. The reliability function of the structure after the failure of the first chain is then reduced to

$$L_1(N) = e^{-10^{-4}N} + \left(1 - e^{-10^{-4}N}\right)e^{-10^{-1}N} \quad (39)$$

Hence, the structure of reduced strength retains a measure of safety comparable to that of the undamaged structure, provided its operating life is suitable reduced. The condition $4 \times 10^{-7} N_0 = 10^{-4} N_1$ furnishes the relation, $N_0/N_1 = 250$, between the operational lives of the undamaged and the damaged structure at the same level of "reliability" $L(N)$. Such a structure, designed for operational life of $N_0 = 10^6$ load applications can, after failure of one-load-path, still be operated for $N_1 = 4 \times 10^3$ load repetitions without reducing its "reliability."

The foregoing analysis illustrates the principle of so-called "fail-safe" design[8] which is quite extensively applied in the design of aircraft structures to eliminate the catastrophic consequences of failure of one-load-path structures. The value N_0/N_1 represents the ratio between design life and maximum length of inspection periods for structures in which failures are not immediately obvious but have to be looked for during inspections.

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8 "Fatigue in Aircraft Structures," by A. M. Freudenthal, Academic Press, New York, N. Y., 1957, p. 308.

The risk of failure associated with the reliability function of a multiple-load-path structure is obtained by applying Eq. 2 to Eq. 38

$$L_0(N).r(N) = P_{F,0}\, e^{-NP_{F,0}}\left(1-e^{-NP_{F,1}}\right)\left(1-e^{-NP_{F,2}}\right)$$

$$+ P_{F,1}\, e^{-NP_{F,1}}\left(1-e^{-NP_{F,2}}\right)\left(1-e^{-NP_{F,0}}\right)$$

$$+P_{F,2}\, e^{-NP_{F,2}}\left(1-e^{-NP_{F,0}}\right)\left(1-e^{-NP_{F,1}}\right) \quad \ldots\ldots\ldots\ldots \quad (40)$$

For long operational lives $N >> 1/P_{F,1} > 1/P_{F,2}$ so that $r(N) \approx P_{F,0}$ the risk of failure is independent of the redundancy of the structure. However, for short lives $1/P_{F,2} < N < 1/P_{F,1}$ the failure risk in first approximation is

$$r(N) \approx P_{F,0}\ \left(1 - e^{-NP_{F,1}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (41a)$$

As lives tend towards zero $\left(0 < N < 1/P_{F,2}\right)$, the risk tends towards zero because

$$r(N) \approx P_{F,0}\, N.P_{F,1}\ \left(1 - e^{-NP_{F,2}}\right) \quad \ldots\ldots\ldots\ldots\ldots (41b)$$

The result clearly illustrates the superiority of the multiple-load-path structure. The risk function approaches zero as the number of load applications tends towards zero, while it approaches the constant value $P_{F,0}$ of the single-load-path-structure only when the number of load applications is no longer small. Thus, for a relatively small number of load applications, the multiple-load-path structure has a much smaller probability of failure and, therefore, a far higher safety than the comparative single-load-path-structure. For structures subject to periodic inspection within intervals associated with these sharply reduced probabilities of failure, the provision of redundancies ensures practically absolute safety.

An m-fold statically indeterminate structure is operationally roughly equivalent to an (m+1) load-path chain structure, in which each chain represents an alternative independent determinate system obtained from the indeterminate system by removing different sets of m redundant members. This multiple-chain structure is subject to (m+1) times the actual load intensity, so that, initially, each chain carries the full load of the structure. As redundants consecutively fail, the load on the remaining redundants increases and so does their probability of failure through consecutive decrease in ν_0. Eqs. 20 to 38 are applicable, with the modification that the values $p_{F1k}, p_{F2k}, \ldots, p_{Fmk}$ may vary widely within each alternative determinate chain-structure.

In the preceding analysis, it has been assumed that failure of one redundant completely eliminates its structural action. In the collapse analysis of m-fold determinate elastic-plastic structures, this is not the case. A different approach to the ultimate-load analysis of such structures is required, for their failure involves the simultaneous attainment of the limiting plastic force or moment in (m+1) members or sections rather than their consecutive elimination. When the loading is proportional and the location of these critical members or sections can be assumed to remain fixed, the probability of failure by plastic collapse (ultimate carrying capacity) is equal to the probability of simultaneous occurrence of (m+1) plastic hinges or members. However, whereas the carrying capacity of each hinge (plastic moment) or of each member

(plastic force) is an independent statistical variable, the applied moments or forces at these hinges or members are not independently variable, but related to the statistical external load by the equations defining the collapse mechanism. These equations arising from the application of the principle of virtual work[9] are usually of the form

$$R = a_1 R_1 + a_2 R_2 + \ldots a_{m+1} R_{m+1} = S \quad \ldots\ldots\ldots \quad (42)$$

in which the coefficients $a_0, a_1 \ldots a_{m+1}$ are determined by the moment- or force-distributions under conditions of collapse. The distribution of R_1 and $R_2 \ldots$ are essentially determined by the distribution of the yield-limit. The distribution of R is then obtained as the distribution of $\sum_{1}^{m+1} a_k R_K$, with the aid of which the probability of failure P_F, under a single load application, is obtained at $P(\nu)_{\nu=1}$, and the survivorship function as $L(N) = \exp\left(-NP_F\right)$.

SUMMARY

The safety factor of a structure is defined in terms of the frequency distribution of the quotient of resistance and load, and its relation with the probability of structural failure is developed. The related concepts of "structural reliability" and of "risk of failure" are introduced and examined as a basis of safety analysis for a finite operational life of non-redundant and of redundant structures.

Methods of safety analysis based on logarithmic-normal and extremal distributions of resistance and of load are developed and the effect of redundancy on the risk and reliability function are examined.

9 "Plastic Analysis of Structures," by P. G. Hodge, McGraw Hill Book Co., Inc., New York, N. Y., 1959, p. 32.

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MATRIX ANALYSIS OF STRUCTURES CURVED IN SPACE

By Frank Baron,[1] F. ASCE

SYNOPSIS

A general solution is given in matrix form for the analysis of a member curved or segmental in space and continuous between two supports. The member may be loaded in any direction.

The solution also yields a general slope deflection equation which involves a stiffness matrix. The stiffness matrix contains all stiffness and carry-over factors for the ends of the member.

INTRODUCTION

A matrix formulation is given for the analysis of a structure which lies in space and which is continuous between two supports (Fig. 1). The structure may have any shape in space and any variation in cross section along its length. It may be subjected to loads in any direction, to moments about any axis, and to linear or angular discontinuities at any given station of the structure. It is assumed that Hooke's Law is applicable and that the rotations and displacements of the structure and small relative to the over-all dimensions of the structure. The definition of a small angle is considered acceptable in the succeeding analyses. Furthermore, the effects of shear deformation and of changes in length of an element of a structure are considered negligible. If desired, adjustments in the succeeding analyses may be made for these effects.

Note.—Discussion open until August 1, 1961. To extend the closing date one month, a written request must be filed with the Executive Secretary, ASCE. This paper is part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 87, No. ST 3, March, 1961.

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The special case of a structure which lies in a plane is included in this formulation.

Signs.—The sign convention is illustrated in Fig. 2. Three orthogonal axes (x, y, and z) are selected for reference, but no part of the structure need lie in the planes defined by these axes. A right-hand rule is used for these and succeeding orthogonal axes. (James P. Michalos, F. ASCE, and the writer have previously used[2] a left-hand rule.) In Fig. 2(a), the structure is loaded with external forces, F, and external moments, M, which are represented as vectors. The components of F and M along the x, y, and z axes are positive when they are against the directions of these axes.

The moments and forces at a section of the structure are designated by

$$\sigma = \begin{bmatrix} m \\ f \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

with subscripts which are explained when introduced. To define positive moments and forces at different sections of the structure, a forward path of an

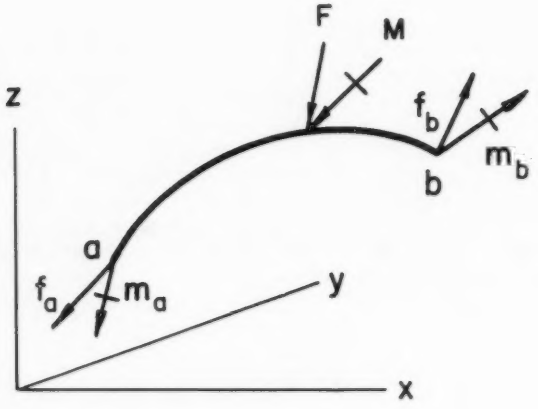

FIG. 1

observer is defined. The forward path of an observer is indicated in Fig. 2 by a curved arrow along the structure. When the observer moves in the direction of this arrow, a section to the back of the observer is a back face and the section to the front is a front face. Forces and moments are positive on a front face when they are in the directions of the axes. They are positive on a back face when they are against the directions of the axes.

The same sign convention is used for rotations, displacements, and angle-changes. In Fig. 2(b), an angle-change or rotation is treated as a force along the axis of rotation. A displacement along an axis is treated as a moment about the axis of displacement. Consequently, computations of geometry involving small angles are interchangeable with those of statics.[2,3] The reciprocal of

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2 "Laterally Loaded Plane Structures and Structures Curved in Space," by Frank Baron and James P. Michalos, Transactions, ASCE, Vol. 117, 1952, pp. 279-311.

3 "Successive Corrections - a Pattern of Thought," by Frank Baron, Numerical Methods of Analysis in Engineering, Chapter 8, L. E. Grinter, Ed., Macmillan Company, 1949.

this interchange is possible, that is, computations of statics can be interchanged with those of a geometry which involves the definition of a small angle.

The structure shown in Fig. 2(b) is the conjugate of that shown in Fig. 2(a). The loads on the conjugate structure are the angle-changes, $d\theta$, of the real

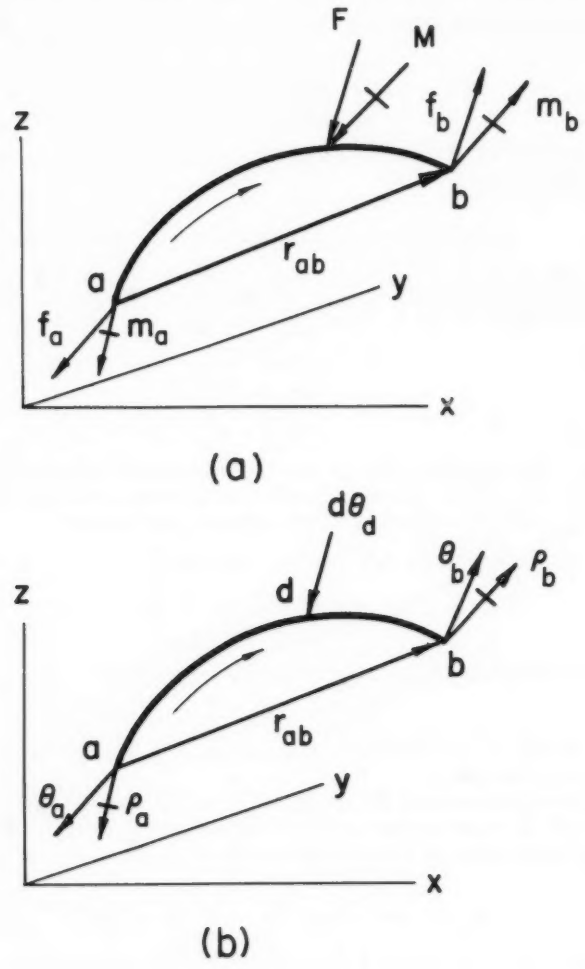


FIG. 2

structure, and the forces and moments at a cross section of the conjugate (or analogous) structure are the same, respectively, as the rotations and displacements of the real structure. The combined rotation and displacement at a sec-

tion of the structure are designed by

$$\Delta = \begin{bmatrix} \theta \\ \rho \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

with subscripts which are explained when introduced.

Position vectors are designated by

$$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Subscripts are used to define the direction of a position vector. For example, r_{ab} is measured from a to b.

PROPERTIES OF AN ELEMENT IN SPACE

An element of a structure in space is shown in Fig. 3. The element is ds in length, is oblique to the x, y, and z axes, and is subjected to a moment of

$$m = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

at each end. The principal axes of the cross sections are orthogonal and are designated by 1, 2, and 3. The unit vectors along these axes are designated by q_1, q_2, and q_3 and are written in matrix notation as follows:

$$q = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} q_{1x} & q_{2x} & q_{3x} \\ q_{1y} & q_{2y} & q_{3y} \\ q_{1z} & q_{2z} & q_{3z} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

in which q_3 is related to q_1 and q_2 by the vector product

$$q_3 = q_1{}^{o}\, q_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

The vector product is denoted by the symbol $q_1{}^{o}$. Eq. 6 holds because the principal axes are orthogonal.

The scalar components of m along the 1, 2, and 3 axes are given by the scalar product $q^*\, m$, in which the asterisk denotes that q is transposed. Consequently, the magnitudes of the angle-changes along the 1, 2, and 3 axes can be written as

$$da_p\, q^*\, m = \begin{bmatrix} da_{11} & 0 & 0 \\ 0 & da_{22} & 0 \\ 0 & 0 & da_{33} \end{bmatrix} q^*\, m \quad \ldots\ldots\ldots\ldots \quad (7)$$

in which the diagonal elements of the da_p matrix are given by

$$da_{11} = \frac{ds}{G\, J_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

$$da_{22} = \frac{ds}{E\, I_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

and

$$da_{33} = \frac{ds}{E\,I_3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (10)$$

The diagonal elements of the da_p matrix are scalar and are the magnitudes of the angle-changes about the 1, 2, and 3 axes caused by moments of one unit about the same axes, respectively. In vector form, the angle-change $d\theta$ caused by the moment m is given by

$$d\theta = q\ da_p\ q^*\ m \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

The angle-change $d\theta$ can also be written as

$$d\theta = \begin{bmatrix} d\theta_x \\ d\theta_y \\ d\theta_z \end{bmatrix} = da\ m = \begin{bmatrix} da_{xx} & da_{xy} & da_{xz} \\ da_{yx} & da_{yy} & da_{yz} \\ da_{zx} & da_{zy} & da_{zz} \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad (12)$$

in which da is the 3 by 3 matrix on the right side of the equation. Each element of the da matrix is an angle-change about an axis designated by the first

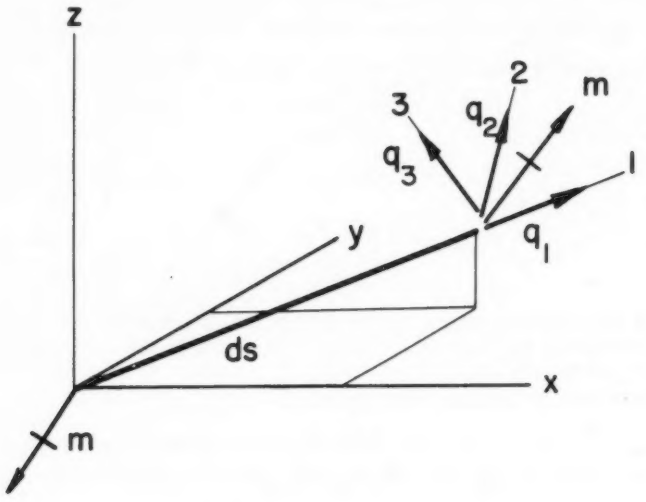


FIG. 3

subscript. The angle-change is caused by a moment of one unit about an axis designated by the second subscript. For example, da_{xy} is an angle-change about the x-axis caused by a moment of one unit about the y-axis. Maxwell's reciprocal relationship holds, and

$$da_{xy} = da_{yx} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

$$da_{yz} = da_{zy} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

and

$$da_{zx} = da_{xz} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

Consequently, da is a symmetric matrix, that is,

$$da = da^* \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

From Eqs. 11 and 12, the following relationship is obtained:

$$da = q\, da_p\, q^* \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

Consequently, each element of the da matrix is defined. For example, the da_{ik} element is given by

$$da_{ik} = q_{1i}\, da_{11}\, q_{1k} + q_{2i}\, da_{22}\, q_{2k} + q_{3i}\, da_{33}\, q_{3k} \quad \ldots\ldots \quad (18)$$

GENERAL STRUCTURE IN SPACE BETWEEN TWO SUPPORTS

Consider the structure shown in Fig. 1. The structure lies in space and is continuous between two supports. It is the same as that dealt with previously.[2] The present formulation differs from the preceding by using matrix notation which lends itself to simplified programming for digital computers.

The analysis of the structure shown in Fig. 1 must fulfill the usual requirements of a statically indeterminate structure. The requirements are as follows:

1. Statics.—For any structure or any part of a structure,

$$\Sigma P = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

and

$$\Sigma r^o P = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

in which

$$r^o = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

Note that the antisymmetric vector matrix r^o contains the components of the vector r in such a way that the product $r^o P$ results in the vector product of the vectors r and P (Fig. 4).

2. Geometry.—For any closed circuit,

$$\Sigma d\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

and

$$\Sigma r^o\, d\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

provided that the angle-changes are small. For these cases, the requirements of geometry are identical with those of statics when an angle-change or rotation is treated as a force, and a displacement is treated as a moment. The preceding statement is the basis for the interchange concept.[2,3]

3. Properties of Materials.—The statical and geometrical relationships must be consistent with the assumed properties of materials. Hooke's Law is assumed.

The structure shown in Fig. 1 may be analyzed by a procedure consisting of two steps as follows:

(1) Assume any statically possible distribution of forces and moments along the structure. Unless this happens to be the correct distribution of forces and moments, errors in geometry will exist.

(2) Subtract a distribution of forces and moments which preserves the requirements of statics but corrects for the errors in geometry.

In matrix form, the procedure consists of

$$\sigma = \sigma_0 - \sigma_i \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

in which σ designates the final moments and forces acting on any section of the structure, and σ_0 and σ_i designate the moments and forces which are associated with steps 1 and 2, respectively. The rotations and displacements at any section of the structure are given by

$$\Delta = \Delta_0 - \Delta_i \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

The procedure automatically satisfies the requirements of statics. It remains to satisfy the requirements of geometry.

The loading conditions for steps 1 and 2 are shown in Fig 5. They are shown for a segment of the structure with an imaginary rigid arm extending from point p at the origin of the x, y, and z-axes to end a of the structure. An

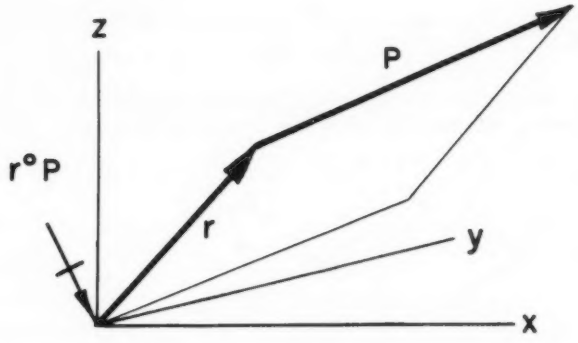


FIG. 4

imaginary rigid arm also extends from point p to end b. These arms are introduced for convenience in closing the circuit. For step 2, statics yields

$$\sigma_i = \begin{bmatrix} I & -r_p^o \\ 0 & I \end{bmatrix} \sigma_{pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (26)$$

in which I is a 3 by 3 identity matrix, 0 is a 3 by 3 null matrix, and r_p^o is an antisymmetric vector matrix containing the x, y, and z components of vector r_p.

The requirements of geometry, given by Eqs. 22 and 23, yield

$$\Sigma \, da \, m_i = \Sigma \, da \left(m_{pi} - r_p^o f_{pi}\right) = \Sigma \, d \, a \, m_o \quad \ldots\ldots\ldots\ldots \quad (27)$$

and

$$\Sigma \, r_p^o \, da \, m_i = \Sigma r_p^o \, da \left(m_{pi} - r_p^o f_{pi}\right) = \Sigma \, r_p^o \, da \, m_o \quad \ldots\ldots\ldots \quad (28)$$

in which the summations are performed for the closed circuit pabp. The terms with the subscript p are referred to the origin at p.

Now let

$$A = \Sigma\, da \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

$$l_p = \Sigma\, r_p^o\, da \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

$$J_p = -\Sigma\, r_p^o\, da\, r_p^o \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

$$P_{po} = \Sigma\, da\, m_o \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

and

$$M_{po} = \Sigma\, r_p^o\, da\, m_o \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (33)$$

and observe that $\left(r_p^o\, da\right)^* = da^*\, r_p^{o*} = -da\, r_p^o$. Therefore,

$$l_p^* = -\Sigma\, da\, r_p^o \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (34)$$

and Eqs. 27 and 28 can be rewritten as

$$A\, m_{pi} + l_p^*\, f_{pi} = P_{po} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

and

$$l_p\, m_{pi} + J_p\, f_{pi} = M_{po} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (36)$$

respectively. A solution for m_{pi} and f_{pi} can be obtained by means of Eqs. 35 and 36 as follows: Rewrite the equations in matrix form as

$$\begin{bmatrix} A & l_p^* \\ l_p & J_p \end{bmatrix} \begin{bmatrix} m_{pi} \\ f_{pi} \end{bmatrix} = \begin{bmatrix} P_{po} \\ M_{po} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (37)$$

and also as

$$K_p\, \sigma_{pi} = \Delta_{po} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (38)$$

in which K_p, σ_{pi}, and Δ_{po} are the first, second, and third matrices of Eq. 37, respectively. Eq. 38 when premultiplied by K_p^{-1} yields

$$\sigma_{pi} = K_p^{-1}\, \Delta_{po} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (39)$$

The solution given by Eq. 39 is complete and involves the inversion of a 6 by 6 matrix. In general, this imposes no serious problems as the inversion can be performed on any computer with a subroutine capable of inverting a 6 by 6 matrix. However, the solution can be obtained in a simpler form which involves the inversion of 3 by 3 matrices instead. For that purpose, premultiply Eq. 35 by $l_p\, A^{-1}$ and subtract the result from Eq. 36, thus obtaining

$$\left(J_p - l_p\, A^{-1}\, l_p^*\right) f_{pi} = \left(M_{po} - l_p\, A^{-1}\, P_{po}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

Let

$$\overline{J}_p = J_p - l_p\, A^{-1}\, l_p^* \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (41)$$

and

$$\overline{M}_{po} = M_{po} - l_p\, A^{-1}\, P_{po} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (42)$$

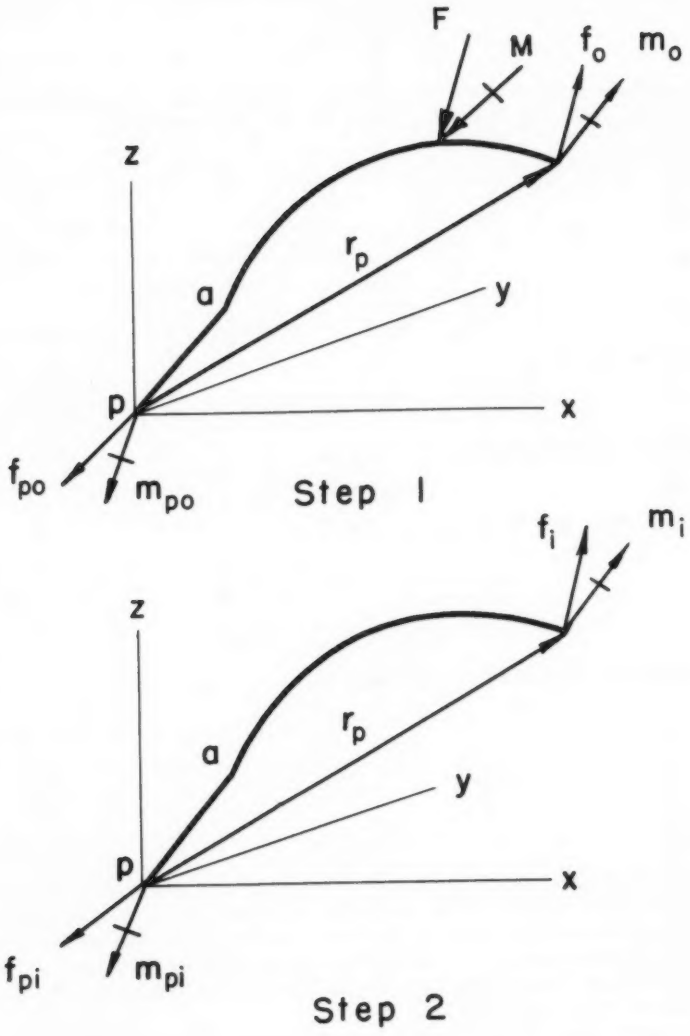


FIG. 5

and rewrite Eq. 40 as

$$\overline{J}_p f_{pi} = \overline{M}_{po} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (43)$$

which when premultiplied by $\overline{J}_p^{-1}$ yields

$$f_{pi} = \overline{J}_p^{-1} \overline{M}_{po} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (44)$$

The solution for m_{pi} is obtained in a similar way. Premultiply Eq. 36 by $l_p^* J_p^{-1}$ and subtract the result from Eq. 35, thus obtaining

$$\left(A - l_p^* J_p^{-1} l_p\right) m_{pi} = \left(P_{po} - l_p^* J_p^{-1} M_{po}\right) \quad \ldots\ldots\ldots \quad (45)$$

Let

$$\overline{A}_p = A - l_p^* J_p^{-1} l_p \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (46)$$

and

$$\overline{P}_{po} = P_{po} - l_p^* J_p^{-1} M_{po} \quad \ldots\ldots\ldots\ldots\ldots \quad (47)$$

and rewrite Eq. 45 as

$$\overline{A}_p m_{pi} = \overline{P}_{po} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (48)$$

which when premultiplied by $\overline{A}_p^{-1}$ yields

$$m_{pi} = \overline{A}_p^{-1} \overline{P}_{po} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (49)$$

Therefore, Eqs. 44 and 49 can be written as

$$\sigma_{pi} = \begin{bmatrix} \overline{A}_p^{-1} & 0 \\ 0 & \overline{J}_p^{-1} \end{bmatrix} \begin{bmatrix} \overline{P}_{po} \\ \overline{M}_{po} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \quad (50)$$

The solution is completed by recalling that

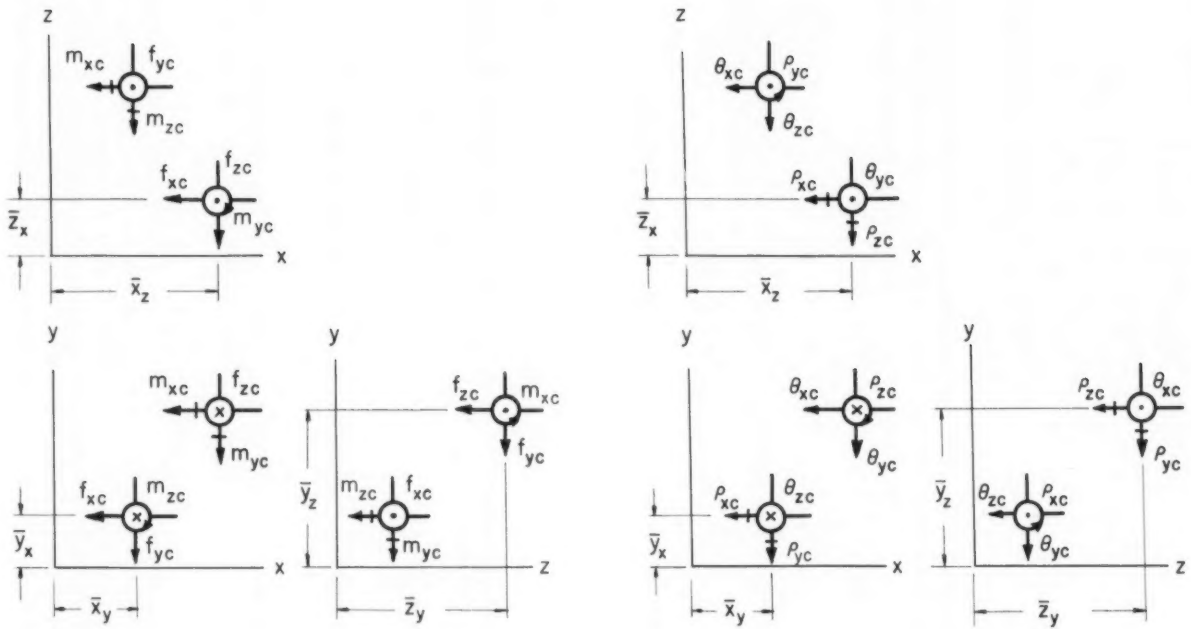
$$\sigma_i = \begin{bmatrix} I & -r_p^o \\ 0 & I \end{bmatrix} \sigma_{pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (51)$$

and

$$\sigma = \sigma_o - \sigma_i \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (52)$$

The preceding analysis is stated in terms of any arbitrarily selected x, y, and z axes. For computational purposes, it is suggested that the position of p be selected somewhere near the average location of the da's of the structure. Such a selection is an informal attempt to diagonalize $A_p\ l_p\ J_p$ and to minimize the sensitivity of succeeding inversions. The formal procedure for doing this is explained in the next section.

Centroidal Axes.—Consider the same structure shown in Fig. 5 but with the imaginary arm now extending from end a to a position that will be referred to as a centroid. For the generalized structure, more than one centroid is required (Fig. 6). In matrix notation, the moments and forces shown in Fig. 6(a)



(a) Statical Relationships

(b) Geometrical Relationships

FIG. 6

are designated by

$$m_{ci} = \begin{bmatrix} m_{xci} \\ m_{yci} \\ m_{zci} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (53)$$

and

$$f_{ci} = \begin{bmatrix} f_{xci} \\ f_{yci} \\ f_{zci} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (54)$$

The forces are along axes which, for convenience, are called centroidal axes. The positions of these axes with reference to the x, y, and z axes are given by

$$\bar{r}_x = \begin{bmatrix} 0 \\ \bar{y}_x \\ \bar{z}_x \end{bmatrix} ; \; \bar{r}_y = \begin{bmatrix} \bar{x}_y \\ 0 \\ \bar{z}_y \end{bmatrix} ; \; \bar{r}_z = \begin{bmatrix} \bar{x}_z \\ \bar{y}_z \\ 0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad (55)$$

Statics of the rigid arm between c and p yields

$$\sigma_{ci} = \begin{bmatrix} I & -\bar{r}^o \\ 0 & I \end{bmatrix} \quad \sigma_{pi} = c_c \, \sigma_{pi} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (56)$$

in which the elements of the $\bar{r}^o$ matrix are arranged as follows:

$$\bar{r}^o = \begin{bmatrix} 0 & -\bar{z}_y & \bar{y}_z \\ \bar{z}_x & 0 & -\bar{x}_z \\ -\bar{y}_x & \bar{x}_y & 0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (57)$$

Note that the matrix product $\bar{r}^o \, f_{ci}$ yields the moment of f_{ci} with respect to an axis through p, and that

$$c_c^{-1} = \begin{bmatrix} I & \bar{r}^o \\ 0 & I \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (58)$$

The correction moments and forces at any cross section of the structure are given by

$$\sigma_i = \begin{bmatrix} I & -r_c^o \\ 0 & I \end{bmatrix} \quad \sigma_{ci} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (59)$$

in which

$$r_c^o = r_p^o - \bar{r}^o \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (60)$$

In Fig. 6(b) rotations about the x, y, and z axes and displacements along the centroidal axes are shown. Each axis of rotation is selected to correspond with that of a correction moment and each axis of displacement corresponds with that of a correction force. Note that the matrix product $\bar{r}^{o*} \, \theta_p$ yields the contribution of a rotation at p to displacements along the centroidal axes.

Therefore, considerations of geometry for the rigid arm yield

$$\Delta_{ci} = \begin{bmatrix} \theta_{ci} \\ \rho_{ci} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \bar{r}^{o*} & I \end{bmatrix} \begin{bmatrix} \theta_{pi} \\ \rho_{pi} \end{bmatrix} = k_c \Delta_{pi} \quad \ldots\ldots\ldots (61)$$

and

$$\Delta_{co} = \begin{bmatrix} P_{co} \\ M_{co} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \bar{r}^{o*} & I \end{bmatrix} \begin{bmatrix} P_{po} \\ M_{po} \end{bmatrix} = k_c \Delta_{po} \quad \ldots\ldots\ldots (62)$$

When desired, the inverse of k_c is given by

$$k_c^{-1} = \begin{bmatrix} I & 0 \\ -\bar{r}^{o*} & I \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (63)$$

The requirements of geometry for a closed circuit

$$\Sigma \, da \, m = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (64)$$

and

$$\Sigma \, r_p^{o} \, da \, m = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (65)$$

can be transformed to the centroidal axes by simple matrix manipulation. From the preceding development,

$$K_p \sigma_{pi} = \Delta_{po} \quad \ldots\ldots\ldots\ldots\ldots\ldots (66)$$

Substituting

$$\sigma_{pi} = c_c^{-1} \sigma_{ci} \quad \ldots\ldots\ldots\ldots\ldots\ldots (67)$$

and

$$\Delta_{po} = k_c^{-1} \Delta_{co} \quad \ldots\ldots\ldots\ldots\ldots\ldots (68)$$

into Eq. 66, and premultiplying the resulting equation by k_c yields

$$k_c K_p c_c^{-1} \sigma_{ci} = \Delta_{co} \quad \ldots\ldots\ldots\ldots\ldots\ldots (69)$$

which can be rewritten as

$$K_c \sigma_{ci} = \Delta_{co} \quad \ldots\ldots\ldots\ldots\ldots\ldots (70)$$

Consequently, the elements of K_c can be obtained from those of K_p by means of

$$K_c = \begin{bmatrix} A & l_c^{*} \\ l_c & J_c \end{bmatrix} = k_c K_p c_c^{-1} \quad \ldots\ldots\ldots\ldots\ldots (71)$$

The centroidal or preferred axes can be located by obtaining the elements of $\bar{r}^{o}$ in Eq. 57. From Eq. 71, we obtain

$$l_c^{*} = A \bar{r}^{o} + l_p^{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots (72)$$

which is of the general form

$$\Lambda = AB + C \quad \ldots\ldots\ldots\ldots\ldots\ldots (73)$$

The elements of $\bar{r}^{o}$ are selected such that l_c^{*} is a diagonal matrix when the diagonal elements of $\bar{r}^{o}$ are zero. Based on these stipulations, the solution for $\bar{r}^{o}$ is given by

$$\bar{r}^{o} = A^{-1} \left(l_c^{*} - l_p^{*} \right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (74)$$

in which the diagonal elements of l_c^* are given by

$$\left(l_c^*\right)_{ii} = \frac{\left(A^{-1}\, l_p^*\right)_{ii}}{\left(A^{-1}\right)_{ii}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (75)$$

In matrix notation, the subscripts ii designate the diagonal elements of a matrix.

With reference to the centroidal axes, the correction moments and forces at c are given by

$$\sigma_{ci} = \begin{bmatrix} \overline{A}_c^{-1} & 0 \\ 0 & \overline{J}_c^{-1} \end{bmatrix} \begin{bmatrix} \overline{P}_{co} \\ \overline{M}_{co} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \quad (76)$$

in which

$$\overline{A}_c = A - l_c^*\, J_c^{-1}\, l_c \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (77)$$

$$\overline{J}_c = J_c - l_c\, A^{-1}\, l_c^* \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (78)$$

$$\overline{P}_{co} = P_{co} - l_c^*\, J_c^{-1}\, M_{co} \quad \ldots\ldots\ldots\ldots\ldots \quad (79)$$

and

$$\overline{M}_{co} = M_{co} - l_c\, A^{-1}\, P_{co} \quad \ldots\ldots\ldots\ldots\ldots \quad (80)$$

The correction moments and forces at any section of the structure are given by Eq. 59.

Other Axes.—Eq. 71 is a general equation that holds for other axes besides the centroidal axes. For example, consider x, y, and z axes that go through a point q at a distance

$$r_q = \begin{bmatrix} x_q \\ y_q \\ z_q \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (81)$$

from point p. For this case, Eq. 71 is transformed into

$$K_q = k_q\, K_p\, k_q^* = \begin{bmatrix} A & l_q^* \\ l_q & J_q \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \quad (82)$$

in which

$$k_q = \begin{bmatrix} I & 0 \\ r_q^{\,o} & I \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (83)$$

and $r_q^{\,o}$ is the antisymmetric matrix,

$$r_q^{\,o} = \begin{bmatrix} 0 & -z_q & y_q \\ z_q & 0 & -x_q \\ -y_q & x_q & 0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \quad (84)$$

Eq. 82 can be used for transferring the computations of K referred to a point p to those referred to a point q. Furthermore, the same equations can be used

for transferring the computations of areas, first moments of areas, and moments of inertias to parallel axes.

STIFFNESS MATRICES

Stiffness and carry-over factors for use in a moment distribution or a slope-deflection method can be obtained for the general member in space. The fac-

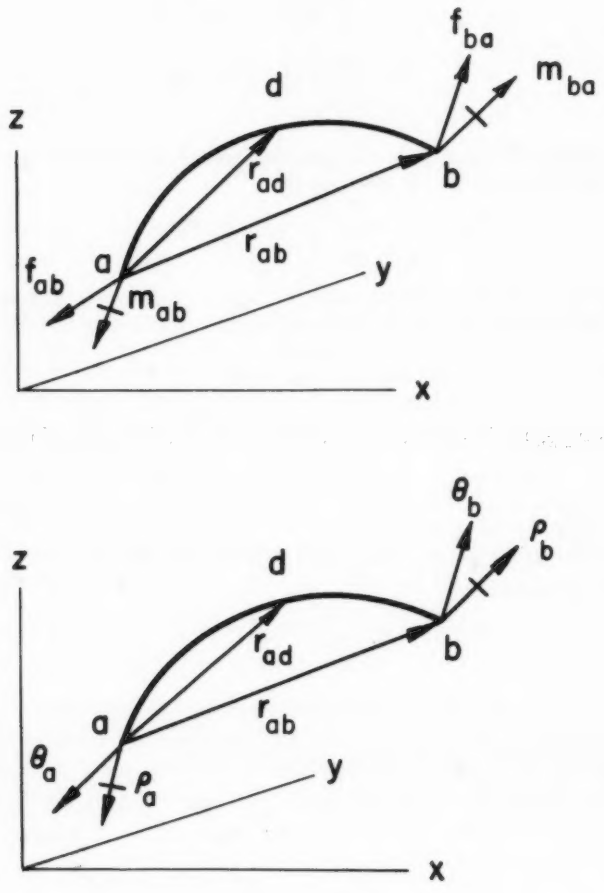


FIG. 7

tors are assembled in matrix form to yield the fundamental slope-deflection relationships for the member.

The member is shown in Fig. 7 and is subjected to slopes and deflections,

$$\Delta_a = \begin{bmatrix} \theta_a \\ \rho_a \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (85)$$

and

$$\Delta_b = \begin{bmatrix} \theta_b \\ \rho_b \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (86)$$

at ends a and b, respectively. No loads are on the member between ends a and b. The moments and forces at ends a and b are designated by

$$\sigma_{ab} = \begin{bmatrix} m_{ab} \\ f_{ab} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (87)$$

and

$$\sigma_{ba} = \begin{bmatrix} m_{ba} \\ f_{ba} \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (88)$$

in which the first subscript of each term designates the end of the member to which the term is referred. Statics yields

$$\sigma_{ba} = \begin{bmatrix} I & -r_{ab}^{o} \\ 0 & I \end{bmatrix} \sigma_{ab} = k_a \sigma_{ab} \quad \ldots\ldots\ldots\ldots \quad (89)$$

in which r_{ab}^{o} is an antisymmetric vector matrix containing the x, y, and z components of vector r_{ab}. Statics of the member between a and d yields

$$\sigma_d = \begin{bmatrix} I & -r_{ad}^{o} \\ 0 & I \end{bmatrix} \sigma_{ab} \quad \ldots\ldots\ldots\ldots\ldots\ldots (90)$$

The requirements of geometry for the member between a and b yield

$$\Sigma\, da\left(m_{ab} - r_{ad}^{o} f_{ab}\right) = \theta_b - \theta_a \quad \ldots\ldots\ldots\ldots \quad (91)$$

and

$$\Sigma\, r_{ad}^{o}\, da\left(m_{ab} - r_{ad}^{o} f_{ab}\right) = \rho_b + r_{ab}^{o}\, \theta_b - \rho_a \quad \ldots\ldots\ldots \quad (92)$$

which can be rewritten as

$$A\, m_{ab} + l_a^{*} f_{ab} = \theta_b - \theta_a \quad \ldots\ldots\ldots\ldots\ldots \quad (93)$$

and

$$l_a\, m_{ab} + J_a f_{ab} = \rho_b + r_{ab}^{o}\, \theta_a - \rho_a \quad \ldots\ldots\ldots\ldots \quad (94)$$

In these equations, A, l_a, and J_a are of the same kind as those given by Eqs. 29 to 31. The l_a, J_a, and l_a^{*} terms however are referred to x, y, and z axes at end a. Eq. 93 and 94 are combined into the single matrix equation

$$K_a \sigma_{ab} = -\Delta_a + k_b^{*} \Delta_b \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (95)$$

in which

$$K_a = \begin{bmatrix} A & l_a^{*} \\ l_a & J_a \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (96)$$

and

$$k_b^{*} = \begin{bmatrix} I & 0 \\ r_{ab}^{o} & I \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (97)$$

The solution for σ_{ab} is given by

$$\sigma_{ab} = - K_a^{-1} \Delta_a + K_a^{-1} k_b^* \Delta_b \quad \ldots\ldots\ldots\ldots \quad (98)$$

which when substituted into Eq. 89 yields

$$\sigma_{ba} = - k_a K_a^{-1} \Delta_a + k_a K_a^{-1} k_b^* \Delta_b \ldots\ldots\ldots \quad (99)$$

Assembling Eqs. 98 and 99 into a single matrix equation yields

$$\overline{\sigma}_{ab} = \overline{E}_{ab} \overline{\Delta}_{ab} \ldots\ldots\ldots\ldots\ldots\ldots \quad (100)$$

in which

$$\overline{\sigma}_{ab} = \begin{bmatrix} \sigma_{ab} \\ \sigma_{ba} \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots \quad (101)$$

$$\overline{\Delta}_{ab} = \begin{bmatrix} \Delta_a \\ \Delta_b \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots \quad (102)$$

and

$$\overline{E}_{ab} = \begin{bmatrix} -K_a^{-1} & K_a^{-1} k_b^* \\ -k_a K_a^{-1} & k_a K_a^{-1} k_b^* \end{bmatrix} \ldots\ldots\ldots\ldots \quad (103)$$

Eq. 100 is the fundamental slope-deflection equation for a general member lying in space. The equation relates the moments and forces at the ends of the member to the slopes and deflections at the ends, by means of the properties of the member. The properties are given by $\overline{E}_{ab}$ which is called the stiffness matrix of the member. The stiffness matrix is 12 by 12 and involves the inversion of a 6 by 6 matrix, namely of K_a.

The solution for $\overline{E}_{ab}$ can be obtained in a simpler form which involves the inversion of 3 by 3 matrices instead. The latter form also clarifies the physical significance of each sub-matrix and element of $\overline{E}_{ab}$. For these purposes, rewrite Eqs. 98 and 99 as

$$\begin{bmatrix} \sigma_{ab} \\ \sigma_{ba} \end{bmatrix} = \begin{bmatrix} E_{a)a} & E_{a)b} \\ E_{b)a} & E_{b)b} \end{bmatrix} \begin{bmatrix} \Delta_a \\ \Delta_b \end{bmatrix} \ldots\ldots\ldots\ldots \quad (104)$$

in which, for example, $E_{a)b}$ designates the moments and forces at a) caused by unit rotations and unit displacements at)b when end a is fixed. Eq. 104 is simply a statement of proportionality between causes and effects for the given member.

In the same way as in the previous section, Eqs. 93 and 94 can be operated on by $l_a A^{-1}$ and $l_a^* J_a^{-1}$ to yield

$$E_{a)a} = \begin{bmatrix} -\overline{A}_a^{-1} & \left(\overline{A}_a^{-1} l_a^* J_a^{-1}\right) \\ \left(J_a^{-1} l_a A^{-1}\right) & -\overline{J}_a^{-1} \end{bmatrix} = K_a^{-1} \ldots\ldots \quad (105)$$

and

$$E_{a)b} = - E_{a)a} k_b^* \ldots\ldots\ldots\ldots\ldots \quad (106)$$

in which $\overline{J}_a$ and $\overline{A}_a$ are the same kind of terms as those given by Eqs. 41 and 46. From Eqs. 89, 104, and 106, we obtain

$$\left[E_{b)a}\ E_{b)b}\right] = \left[k_a\ E_{a)a} - k_a\ E_{a)a}\ k_b^*\right] \ldots\ldots\ldots \quad (107)$$

which completes the solution for $\overline{E}_{ab}$ based on the inverses of 3 by 3 matrices instead of a 6 by 6 matrix.

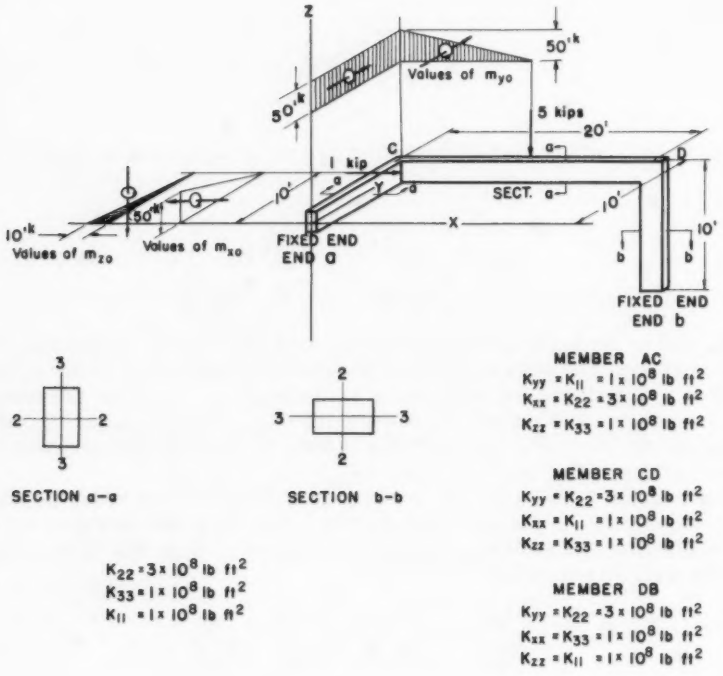


FIG. 8

Stiffness and Carry-Over Factors.—For use in a moment distribution method, E_{ab} can be partitioned to yield the stiffness factors and carry-over factors for the ends of the member. The partitioning is as follows:

$$\overline{E}_{ab} = \left[\begin{array}{c|c} E_{a)a} & E_{a)b} \\ \hline E_{b)a} & E_{b)b} \end{array}\right] = \left[\begin{array}{cc|cc} S_{a)\,a}^{r} & S_{a)\,a}^{t} & S_{a)\,b}^{r} & S_{a)\,b}^{t} \\ K_{a)\,a}^{r} & K_{a)\,a}^{t} & K_{a)\,b}^{r} & K_{a)\,b}^{t} \\ \hline S_{b)\,a}^{r} & S_{b)\,a}^{t} & S_{b)\,b}^{r} & S_{b)\,b}^{t} \\ K_{b)\,a}^{r} & K_{b)\,a}^{t} & K_{b)\,b}^{r} & K_{b)\,b}^{t} \end{array}\right] \ldots\ldots \quad (108)$$

in which each S and K term can be further partitioned, for example, as follows:

$$S_{a)\,a}^{r} = \begin{bmatrix} S_{ax)\,a}^{rx} & S_{ax)\,a}^{ry} & S_{ax)\,a}^{rz} \\ S_{ay)\,a}^{rx} & S_{ay)\,a}^{ry} & S_{ay)\,a}^{rz} \\ S_{az)\,a}^{rx} & S_{az)\,a}^{ry} & S_{ay)\,a}^{rz} \end{bmatrix} \quad \ldots\ldots\ldots\ldots \quad (109)$$

In Eqs. 108 and 109, the S terms designate moments and the K terms designate forces at the ends of the member. The subscripts are read as follows: $S_{a)\,b}^{r}$ represents the moments at a) caused by unit rotations at b when end a is fixed. $S_{ay)\,a}^{rx}$ represents the moment about the y axis at a caused by a unit rotation about the x axis at a, with no other rotation or translation occurring at the ends of the member.

In brief, the elements of $\overline{E}_{ab}$ represent stiffness factors and products of stiffness factors and carry-over factors as defined in moment distribution.

Reciprocal Relationships.—Much symmetry exists in the construction of $\overline{E}_{ab}$ because of Maxwell's reciprocal relationships for elastic members. For example,

$$S_{a)\,a}^{r} = S_{a)\,a}^{r\,*} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (110)$$

$$K_{a)\,a}^{t} = K_{a)\,a}^{t\,*} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (111)$$

$$K_{a)\,a}^{r} = S_{a)\,a}^{t\,*} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (112)$$

$$S_{a)\,b}^{r} = -S_{b)\,a}^{r\,*} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (113)$$

$$K_{a)\,b}^{t} = -K_{b)\,a}^{t\,*} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (114)$$

and

$$K_{a)\,b}^{r} = -S_{b)\,a}^{t\,*} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (115)$$

In this instance, Maxwell's reciprocal relationships are for reactions caused by unit displacements. Note that the reciprocal reactions must correspond with the unit displacements in accordance with the definition of work. For this reason,

$$K_{a)\,a}^{r} \neq K_{a)\,a}^{r\,*} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (116)$$

TABLE 1

Matrix			Equation	Matrix			Equation	Matrix			Equation
A	$=10^{-9}$	$\begin{bmatrix} 333.3 & 0 & 0 \\ 0 & 200. & 0 \\ 0 & 0 & 400. \end{bmatrix}$	29	$l_a A^{-1} l_a^*$	$=10^{-7}$	$\begin{bmatrix} 308. & -350. & 11.1 \\ -350. & 408. & 47.5 \\ 11.1 & 47.5 & 390. \end{bmatrix}$	41	P_{ao}	$=10^{-7}$	$\begin{bmatrix} -8.33 \\ 58.3 \\ -5.00 \end{bmatrix}$	32
l_a	$=10^{-9}$	$\begin{bmatrix} 0 & 166.5 & 3500 \\ -500 & 0 & -4000 \\ -3167. & 1332. & 0 \end{bmatrix}$	30	$l_a^* J_a^{-1} l_a$	$=10^{-9}$	$\begin{bmatrix} 189. & -78.7 & 20.4 \\ -78.7 & 33.5 & 8.36 \\ 20.4 & 8.36 & 358. \end{bmatrix}$	46	M_{ao}	$=10^{-7}$	$\begin{bmatrix} -16.6 \\ 0 \\ 55.4 \end{bmatrix}$	33
l_a^*	$=10^{-9}$	$\begin{bmatrix} 0 & -500 & -3167 \\ 166.5 & 0 & 1332 \\ 3500 & -4000 & 0 \end{bmatrix}$	34	$\bar{A}_a$	$=10^{-9}$	$\begin{bmatrix} 144. & 78.7 & -20.4 \\ 78.7 & 167. & -8.36 \\ -20.4 & -8.36 & 41.7 \end{bmatrix}$	46	$l_a A^{-1} P_{ao}$	$=10^{-7}$	$\begin{bmatrix} 4.86 \\ 62.5 \\ 468. \end{bmatrix}$	42
J_a	$=10^{-7}$	$\begin{bmatrix} 344.4 & -400 & 33.3 \\ -400. & 700 & 50.0 \\ 33.3 & 50.0 & 533. \end{bmatrix}$	31	$\bar{J}_a$	$=10^{-7}$	$\begin{bmatrix} 36.8 & -50.0 & 22.2 \\ -50.0 & 293. & 2.5 \\ 22.2 & 2.5 & 143. \end{bmatrix}$	41	$l_a^* J_a^{-1} M_{ao}$	$=10^{-9}$	$\begin{bmatrix} -345. \\ 137. \\ -221. \end{bmatrix}$	47
A^{-1}	$=10^{9}$	$\begin{bmatrix} .003 & 0 & 0 \\ 0 & .005 & 0 \\ 0 & 0 & .0025 \end{bmatrix}$	41, 42	$\bar{A}_a^{-1}$	$=10^{6}$	$\begin{bmatrix} 9.97 & -4.51 & 3.97 \\ -4.51 & 8.11 & -.579 \\ 3.97 & -.579 & 25.8 \end{bmatrix}$	49	$\bar{P}_{ao}$	$=10^{-7}$	$\begin{bmatrix} -4.88 \\ 57.0 \\ -2.79 \end{bmatrix}$	47
J_a^{-1}	$=10^{5}$	$\begin{bmatrix} .921 & .534 & -.108 \\ .534 & .453 & -.0759 \\ -.108 & -.0759 & .201 \end{bmatrix}$	46, 47	$\bar{J}_a^{-1}$	$=10^{5}$	$\begin{bmatrix} 4.05 & .698 & -.639 \\ .698 & .462 & -.116 \\ -.639 & -.116 & .798 \end{bmatrix}$	44	$\bar{M}_{ao}$	$=10^{-7}$	$\begin{bmatrix} -21.5 \\ -62.5 \\ -413. \end{bmatrix}$	42
$m_{ai} = \begin{bmatrix} -31.7 \\ 48.7 \\ -12.4 \end{bmatrix}$ ft kips; $f_{ai} = \begin{bmatrix} 1.33 \\ 0.04 \\ -3.08 \end{bmatrix}$ kips											50

and

$$S_{a)t}{}_{a} \neq S_{a)t}{}_{a}^{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (117)$$

NUMERICAL EXAMPLE

Fig. 8 shows a structure consisting of three prismatic members, each of which is parallel to a rectangular axis. The structure is fixed at ends a and b and is subjected to two concentrated loads. The assumed distributions of moments and forces are for a structure cantilevered from end a. The structure

TABLE 2

Matrix			Equation	Matrix			Equation
$A^{-1}l_a^* = 10^0$	$\begin{bmatrix} 0 & -1.50 & -9.50 \\ .833 & 0 & 6.66 \\ 8.75 & -10.0 & 0 \end{bmatrix}$		75	$\bar{A}_c^{-1} = 10^9$	$\begin{bmatrix} .003 & 0 & 0 \\ 0 & .005 & 0 \\ 0 & 0 & .0025 \end{bmatrix}$		76
$l_c^* = 10^0$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$		75	$\bar{J}_c^{-1} = 10^5$	$\begin{bmatrix} 4.05 & .698 & -.639 \\ .698 & .462 & -.116 \\ -.639 & -.116 & .698 \end{bmatrix}$		76
$\bar{r}^0 = 10^0$	$\begin{bmatrix} 0 & 1.50 & 9.50 \\ -.833 & 0 & -6.66 \\ -8.75 & 10.0 & 0 \end{bmatrix}$		57,74	$P_{co} = 10^{-7}$	$\begin{bmatrix} -.833 \\ 58.3 \\ -5.0 \end{bmatrix}$		62
$J_c = 10^{-7}$	$\begin{bmatrix} 36.8 & -50. & 22.2 \\ -50. & 293. & 2.5 \\ 22.2 & 2.5 & 143. \end{bmatrix}$		71	$M_{co} = 10^{-7}$	$\begin{bmatrix} -21.4 \\ -62.5 \\ -412. \end{bmatrix}$		62
$\bar{A}_c = 10^{-9}$	$\begin{bmatrix} 333. & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 400 \end{bmatrix}$		77	$\bar{P}_{co} = 10^{-7}$	$\begin{bmatrix} -8.33 \\ 58.3 \\ -5.0 \end{bmatrix}$		79
$\bar{J}_c = 10^{-7}$	$\begin{bmatrix} 36.8 & -50. & 22.2 \\ -50. & 293. & 2.5 \\ 22.2 & 2.5 & 143. \end{bmatrix}$		78	$\bar{M}_{co} = 10^{-7}$	$\begin{bmatrix} -21.5 \\ -62.5 \\ -413. \end{bmatrix}$		80
$m_{ci} = \begin{bmatrix} -2.50 \\ 29.2 \\ -1.25 \end{bmatrix}$ ft. kips;				$f_{ci} \begin{bmatrix} 1.33 \\ 0.04 \\ -3.08 \end{bmatrix}$ kips			76

and loads are the same as those considered in Figs. 10 and 11 of a previous work[2] by the writer and Michalos.

The matrix analysis of the structure is summarized in Tables 1, 2, and 3. Each matrix in these tables is referred to a corresponding equation in the text. The matrices in Table 1 are for determining the correction forces and moments at any section of the structure when the origin of the x, y, and z axes are at end a. The magnitudes of A, l_a, J_a, P_{ao}, and M_{ao} are the same as those previously given.[2] The signs however are different for several elements of these matrices, because a right hand rule of axes is used herein whereas a left hand

rule was previously used by the writer.[2] In Table 1, the correction moments and forces at end a are obtained.

The matrices in Table 2 are referred to the centroidal axes of the structure. In this table, the correction moments and forces at the centroidal axes are obtained.

TABLE 3

	Matrix	Equation
$K_a = 10^{-7}$	$\begin{bmatrix} 3.33 & 0 & 0 & 0 & -5.00 & -31.7 \\ 0 & 2.0 & 0 & 1.67 & 0 & 13.3 \\ 0 & 0 & 4.0 & 35.0 & -40.0 & 0 \\ 0 & 1.67 & 35.0 & 344. & -400 & 33.3 \\ -5.0 & 0 & -40.0 & -400. & 700 & 50.0 \\ -31.7 & 13.3 & 0 & 33.3 & 50.0 & 533. \end{bmatrix}$	38
$K_c = 10^{-7}$	$\begin{bmatrix} 3.33 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36.8 & -50. & 22.2 \\ 0 & 0 & 0 & -50 & 293. & 2.50 \\ 0 & 0 & 0 & 22.2 & 2.5 & 143. \end{bmatrix}$	70, 71
$E_{a)a} = 10^{5}$	$\begin{bmatrix} -99.7 & 45.1 & -39.7 & 5.03 & .410 & -7.41 \\ 45.1 & -81.1 & 5.79 & -.886 & -.192 & 4.79 \\ -39.7 & 5.79 & -258. & 28.5 & 1.49 & -4.43 \\ 5.03 & -.886 & 28.5 & 4.05 & -.698 & .639 \\ .410 & -.192 & 1.49 & -.698 & -.462 & .116 \\ -7.41 & 4.79 & -4.43 & .639 & .116 & -.798 \end{bmatrix}$	105
$r_{ab}^{o} =$	$\begin{bmatrix} 0 & 10 & 10 \\ -10 & 0 & -20 \\ -10 & 20 & 0 \end{bmatrix}$	89, 97 106,107

Matrices r_{ab}^{o} and $E_{a)a}$ are given in Table 3 for obtaining the stiffness matrix, $\overline{E}_{ab}$, of the structure. The remaining elements of $\overline{E}_{ab}$ can be obtained by means of r_{ab}^{o}, $E_{a)a}$, and Eqs. 104 through 107.

CONCLUDING

The present development has been extended to the matrix analysis of a generalized structure which lies in space, and which is continuous over several rigid or flexible supports.

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DISCUSSION

Note.—This paper is a part of the copyrighted Journal of the Structural Division, Proceedings of the American Society of Civil Engineers, Vol. 87, No. ST 3, March, 1961.

ANALYSIS OF CONTINUOUS TRUSSES BY CARRY-OVER MOMENTS[a]

Closure by Jan J. Tuma

JAN J. TUMA,[1] F. ASCE.—The writer appreciates the discussion of Mr. Petcu, who pointed out the rapid convergence of the carry-over procedure and analyzed the conditions of convergency. The writer disagrees with Mr. Petcu's statement, "that the carry-over procedure does not allow one to obtain a solution with free terms appearing literally and that each load conditions must be investigated separately." The carry-over procedure may be performed in numerical (Part 4 and 5) or algebraic form (Part 6).

The algebraic form is perfectly general in terms of algebraic carry-over factors and starting values. The general solution of three and four span trusses is recorded in Tables 6 and 7. The solution of five and six span trusses is recorded in Tables 18 and 19.

The special method[2] suggested by Mr. Petcu is one of many methods devised for the solution of three-moment equations. Its limitation seems to be the requirement of equal spans.

The writer is indebted to J. T. Oden, A. M. ASCE, who prepared the comparative study of a four span truss by the moment distribution method and the carry-over moment method (Tables 20 and 21). Oden's examples show the advantages and the simplicity of the carry-over moment method.

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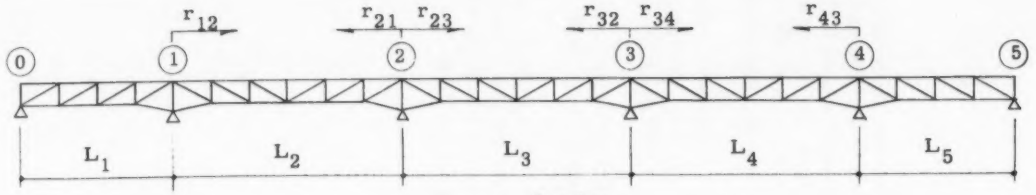
[a] December 1959, by Jan J. Tuma (Proc. Paper 2314)

[1] Prof. of Civ. Engrg., Oklahoma State Univ., Stillwater, Okla.

[2] "Studii Si Cercetari De Mechanica Aplicata," by Valeriu Petcu, Bucharest, No. 3, October, 1959, pp. 855-865.

TABLE 18.—FIVE SPAN TRUSS

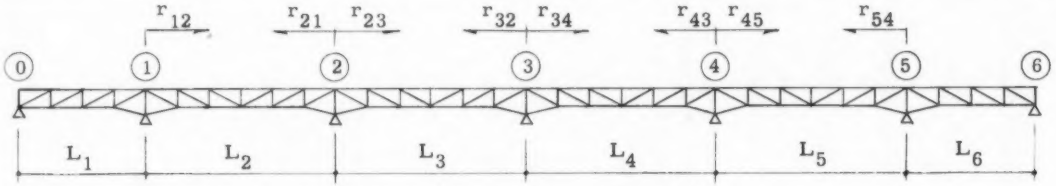


$$X_{12} = 1 - r_{12}r_{21} \qquad Z_{14} = 1 - \frac{r_{23}\,r_{32}}{X_{12}X_{34}} \qquad X_{34} = 1 - r_{34}r_{43}$$

Final Moments

Moment	m_1	m_2	m_3	m_4
M_1	$1 + \frac{r_{12}\,r_{21}}{X_{12}Z_{14}}$	$\frac{r_{21}}{X_{12}Z_{14}}$	$\frac{r_{21}}{r_{23}}\left(\frac{1}{Z_{14}} - 1\right)$	$\frac{r_{21}\,r_{43}}{r_{23}}\left(\frac{1}{Z_{14}} - 1\right)$
M_2	$\frac{r_{12}}{X_{12}Z_{14}}$	$\frac{1}{X_{12}Z_{14}}$	$\frac{1}{r_{23}}\left(\frac{1}{Z_{14}} - 1\right)$	$\frac{r_{43}}{r_{23}}\left(\frac{1}{Z_{14}} - 1\right)$
M_3	$\frac{r_{12}}{r_{32}}\left(\frac{1}{Z_{14}} - 1\right)$	$\frac{1}{r_{32}}\left(\frac{1}{Z_{14}} - 1\right)$	$\frac{1}{X_{34}Z_{14}}$	$\frac{r_{43}}{X_{34}Z_{14}}$
M_4	$\frac{r_{12}\,r_{34}}{r_{32}}\left(\frac{1}{Z_{14}} - 1\right)$	$\frac{r_{34}}{r_{32}}\left(\frac{1}{Z_{14}} - 1\right)$	$\frac{r_{34}}{X_{34}Z_{14}}$	$1 + \frac{r_{34}\,r_{43}}{X_{34}Z_{14}}$

TABLE 19.—SIX SPAN TRUSS



$X_{12} = 1 - r_{12}\,r_{21}$ | $Z_{24} = 1 - \frac{r_{23}r_{32}}{X_{12}} - \frac{r_{34}r_{43}}{X_{45}}$ | $X_{45} = 1 - r_{45}r_{54}$

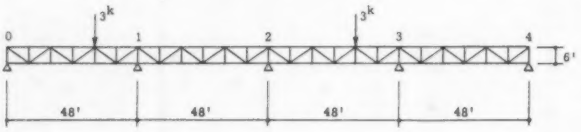
Final Moments

	m_1	m_2	m_3	m_4	m_5
M_1	$\frac{1}{X_{12}}\left(1+\frac{r_{12}r_{21}r_{23}r_{32}}{X_{12}Z_{24}}\right)$	$\frac{r_{21}}{X_{12}}\left(1+\frac{r_{23}r_{32}}{X_{12}Z_{24}}\right)$	$\frac{r_{21}\,r_{32}}{X_{12}Z_{24}}$	$\frac{r_{21}r_{32}r_{43}}{X_{12}X_{45}Z_{24}}$	$\frac{r_{21}r_{32}r_{43}r_{54}}{X_{12}X_{45}Z_{24}}$
M_2	$\frac{r_{12}}{X_{12}}\left(1+\frac{r_{23}\,r_{32}}{X_{12}Z_{24}}\right)$	$\frac{1}{X_{12}}\left(1+\frac{r_{23}\,r_{32}}{X_{12}Z_{24}}\right)$	$\frac{r_{32}}{X_{12}Z_{24}}$	$\frac{r_{32}\,r_{43}}{X_{12}X_{45}Z_{24}}$	$\frac{r_{32}r_{43}r_{54}}{X_{12}X_{45}Z_{24}}$
M_3	$\frac{r_{12}\,r_{23}}{X_{12}Z_{24}}$	$\frac{r_{23}}{X_{12}Z_{24}}$	$\frac{1}{Z_{24}}$	$\frac{r_{43}}{X_{45}Z_{24}}$	$\frac{r_{43}\,r_{54}}{X_{45}Z_{24}}$
M_4	$\frac{r_{12}r_{23}r_{34}}{X_{12}X_{45}Z_{24}}$	$\frac{r_{23}r_{34}}{X_{12}X_{45}Z_{24}}$	$\frac{r_{34}}{X_{45}Z_{24}}$	$\frac{1}{X_{45}}\left(1+\frac{r_{34}\,r_{43}}{X_{45}Z_{24}}\right)$	$\frac{r_{54}}{X_{45}}\left(1+\frac{r_{34}\,r_{43}}{X_{45}Z_{24}}\right)$
M_5	$\frac{r_{12}r_{23}r_{34}r_{45}}{X_{12}X_{45}Z_{24}}$	$\frac{r_{23}r_{34}r_{45}}{X_{12}X_{45}X_{24}}$	$\frac{r_{34}\,r_{45}}{X_{12}X_{45}Z_{24}}$	$\frac{r_{45}}{X_{45}}\left(1+\frac{r_{34}\,r_{43}}{X_{45}Z_{24}}\right)$	$\frac{1}{X_{45}}\left(1+\frac{r_{34}r_{43}r_{45}r_{54}}{X_{45}Z_{24}}\right)$

TABLE 20.—FOUR SPAN TRUSS, MOMENT DISTRIBUTION

Analysis of a Four Span Continuous Truss by the Moment Distribution Method.

λ_m & d_m



(1) Angular Functions (Table 1):

$F_{10} = F_{12} = F_{21} = F_{23} = F_{32} = F_{34} = .978$

$G_{10} = G_{12} = G_{21} = G_{23} = G_{32} = G_{43} = .355$

$\tau_{10} = \tau_{32} = 23.575 \qquad \tau_{23} = 19.089$

(2) Moment Distribution Functions:

(a) Stiffness Factors:

$N = F_{ij} F_{ji} - G_{ij} G_{ji}$

$= (.978)^2 - (.355)^2 = .831$

$K_{12} = K_{21} = K_{23} = K_{32}$

$= \frac{F}{N} = \frac{.978}{.832} = 1.177$

$K'_{10} = K'_{34} = \frac{1}{F} = \frac{1}{.978} = 1.022$

(b) Carry-Over Factors:

$C_{12} = C_{21} = C_{23} = C_{32} = \frac{G}{F} = \frac{.355}{.978} = .363$

(c) Fixed End Moments:

$FM'_{10} = \frac{\tau_{10}}{F_{10}} = + \frac{23.575}{.978} = + 24.10^{k'}$

$FM_{23} = - K_{23} \tau_{23} + C_{32} K_{32} \tau_{32}$

$= (1.177)\ 19.089 - (.363)(23.575)$

$= - 12.37^{k'}$

$FM_{32} = K_{32} \tau_{32} - C_{23} K_{23} \tau_{23}$

$= (1.177)\ 23.575 - (.363)(19.089)$

$= + 19.57^{k'}$

(d) Distribution Factors:

$\Sigma K'_1 = \Sigma K'_3 = K'_{10} + K_{12} = 2.199$

$\Sigma K_2 = \quad = K_{21} + K_{23} = 2.354$

$D_{10} = D_{34} = - \frac{K'_{10}}{\Sigma K'_1} = - .465$

$D_{12} = D_{23} = - \frac{K_{12}}{\Sigma K'_1} = - .535$

$D_{21} = D_{23} = - \frac{K_{21}}{\Sigma K_2} = - .800$

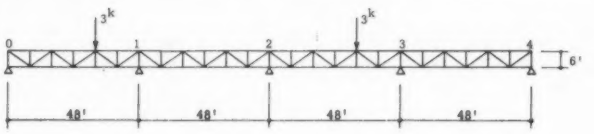
(3) Distribution Procedure:

		M	10	12	① 21	23	② 32	34	③
		D	- .465	- .535	- .500	- .500	- .535	- .465	
		C	+ .000	+ .363	+ .363	+ .363	+ .363	+ .000	
1		FM	+24.10			-12.37	+19.57		
		D	-11.21	-12.89			-10.47	- 9.10	
2		C			- 4.68	- 3.80			
		D			+10.43	+10.42			
3		C		+ 3.78			+ 3.78		
		D	- 1.76	- 2.02			- 2.02	- 1.76	
4		C			- .73	- .73			
		D			+ .73	+ .73			
5		C		+ .27			+ .27		
		D	- .13	- .14			- .14	- .13	
6		C			- .05	- .05			
		D			+ .05	+ .05			
7		C		+ .02			+ .02		
		D	- .01	- .01			- .01	- .01	
Σ			+10.99	-10.99	+ 5.75	- 5.75	+11.00	-11.00	

TABLE 21.—FOUR SPAN TRUSS, CARRY-OVER MOMENT

Analysis of a Four Span Continuous Truss by the Carry-Over Moment Procedure.

λ_m & d_m



(1) Angular Functions (Table 1):

$F_{10} = F_{12} = F_{21} = F_{23} = F_{32} = F_{34} = 0.978$

$G_{10} = G_{12} = G_{21} = G_{23} = G_{32} = G_{34} = 0.355$

$\tau_{10} = \tau_{32} = 23.575 \qquad \tau_{23} = 19.089$

(2) Moment Carry-Over Functions:

(a) Carry-Over Factors:

$\Sigma F_1 = \Sigma F_2 = \Sigma F_3 = F_{10} + F_{12} = 1.956$

$r_{12} = r_{21} = r_{23} = r_{32} = -\frac{.355}{1.956} = -.181$

(b) Starting Moments:

$m_1 = -\frac{\tau_{10}}{\Sigma F_1} = -\frac{23.575}{1.956} = -12.05$

$m_2 = -\frac{\tau_{23}}{\Sigma F_2} = -\frac{19.089}{1.956} = -9.76$

$m_3 = -\frac{\tau_{32}}{\Sigma F_3} = -\frac{23.575}{1.956} = -12.05$

(3) Carry-Over Procedure

		①	②	③
	r	$\overrightarrow{-.181}$	$\overleftarrow{-.181}$	$\overrightarrow{-.181}$
	m	-12.05	-9.76	-12.05
1	C.O.		+2.19 +2.19	
2	C.O.	+.97		+.97
3	C.O.		-.18 -.18	
4	C.O.	+.07		+.07
5	C.O.		-.01 -.01	
	Σ	-11.01	-5.76	-11.01

CONTINUOUS GIRDER BRIDGE WITH VARIABLE MOMENT OF INERTIA[a]

Closure by Sabri Sami

SABRI SAMI,[11] F. ASCE.—Those who discussed this paper have presented valuable graphical and analytical alternate procedures. The discussions have added to the value of the paper and the kind comments are appreciated.

The information presented by Mr. Hayashi, A. M. ASCE, using the stiffness matrix analysis is very interesting; actually the two methods are based on the same principle. The main object of the writer is to present a simple method so that any civil engineer can analyze easily and rapidly a continuous bridge with variable moment of inertia.

The writer fails to agree with Mr. Khachaturian that the results of the study, in the form presented by the writer, are of academic interest only or are applicable in a very limited way. The writer believes that the paper presents a practical and simple procedure for the analysis of these bridges so that the designer will be able to study a structure with different relative dimensions using the simple formulas presented and thus choose the most economical dimensions. The assumptions used by the writer are clear and need not be repeated again; I-d (moment of inertia, depth) curves can be drawn and a tentative design can be made for non-rectangular sections using the formulas of the redundants presented in the paper.

The form of expressing the elastic equation of the redundant reaction in the two-span bridge is similar to the approach of Mr. Khachaturian,[6] and virtual work was used in both studies. It should be noted that it was Mr. Khachaturian's encouragement that lead to the attempt to analyze the three-span bridge using the same approach. The writer expanded the method to more than three spans and to different rigid-frame bridges at a later date.

Mr. A. A. Eremin, M. ASCE, has presented a graphical method of moment distribution. The writer is grateful to Mr. Eremin for renewing interest in this interesting method. Fixed points can be found easily by using the analytical results presented in the paper.

For example, in the three-span bridge, case b, Fig. 8, dividing Eq. 10b by Eq. 10a one gets

$$\frac{M_C}{M_B} = \frac{4\,n\,D_1}{Z_3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (62)$$

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[a] January 1960, by Sabri Sami (Proc. Paper 2346).

[11] Prof. of Structural Engrg., West Virginia Univ., Morgantown, West Virginia.

This result can be checked also from case (d) by dividing Eq. 13b by Eq. 13a as follows:

$$\frac{M_C}{M_B} = \frac{4\,n\,D_1\,D_7\,K_3}{D_7\,Z_3\,K_3} = \frac{4\,n\,D_1}{Z_3} \quad \ldots\ldots\ldots\ldots\ldots (63)$$

Therefore, two fixed points (because of symmetry) can be located on Span BC.

In the four-span bridge, case (a), Fig. 13, dividing Eq. 15b by Eq. 15a, yields

$$\frac{M_C}{M_B} = \frac{2\,D_1\,Z_4}{K_4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (64)$$

The same results that can be found by dividing Eq. 21b by Eq. 21a, case (d), Fig. 16. The minus sign has no significance as values of Z_4 are always negative and K_4-values are always positive. Therefore, a fixed point can be located on each of spans BC and CD.

Dividing Eq. 15c by Eq. 15b, case (a), Fig. 13 or Eq. 21c by Eq. 21b, case (d), Fig. 16, the following is obtained:

$$\frac{M_D}{M_C} = -\frac{4\,n\,D_1}{Z_4} \quad \ldots\ldots\ldots\ldots\ldots\ldots (65)$$

The same result is obtained by dividing Eq. 19c by Eq. 19b in case (b), Fig. 14 or dividing Eq. 22c by Eq. 22b, case (e), Fig. 17. This locates another fixed point on each of Spans BC and CD.

A set of fixed point is used in case the moment is applied from the left or from the right. The fixed points, one in each end span and two in each intermediate span, are independent of the loading and are fixed in location for any given bridge with different relative proportions. These fixed points can be used, therefore, to reduce the analytical work or to encourage a graphical method. They indicate also the load position for maximum stresses.

A comparison between different straight or parabolic haunches can be studied through the fixed-point approach. This is being done by the writer and his students. The choice of a graphical or a semi-graphic solution depends on the engineer's general attitude towards graphical solutions.

Mr. Leontovich has shown another approach to the problem using a method that is based on the concept of elastic parameters. There are many approaches to the problem published in the United States and abroad; the writer's method is one of them. The writer believes that his method, as Mr. Leontovich stated, is a "ready-to use" solution for determining the redundants in continuous girders of parabolic shape. It is presented in a manner that can be used easily and rapidly with the use of an ordinary office calculator. The writer is aware of Mr. Leontovich's work and his valuable and important general method.

SECONDARY STRESSES IN PARALLEL WIRE SUSPENSION CABLES[a]

Discussion by T. Janiszewski and Blair Birdsall

T. JANISZEWSKI,[2] and BLAIR BIRDSALL,[3] F. ASCE.—The writers wish to compliment the author on the thorough-going attack he has made on a subject that has had too little attention. After a few tests made in 1922 and 1923 in preparation for the Philadelphia-Camden Bridge that produced some empirical results, designers have been content to assume that the subject secondary stresses were sufficiently inconsequential to require no more attention than a little judgment in setting the factor of safety for a bridge cable. Thus, a thorough analytical study is very much in order as a means of either confirming or revising these empirical conclusions.

Unfortunately, the press of every-day events have denied the writers the pleasure and satisfaction of a full review and detailed mathematical critique of this paper. However, they hope that the few remarks they have been able to piece together will stimulate the author to close the discussion by correlating his mathematical speculations with the results of known tests and with experience.

For the benefit of those who are no more mathematically sophisticated than the writers, they offer the following description of their struggle to overcome an apparent paradox in the text:

In dealing with beam problems, one is accustomed to associating greater stiffness with greater unit stresses for a given deflection. Thus, the writers were puzzled when, in studying Eqs. 4, 5, and 6 they found that, according to these expressions, unit secondary stress (Eq. 6) increases as stiffness decreases. To clarify, note that G, although defined only as "flexural stiffness of one panel of cable," is applied mathematically as being equal to $E\,I/l_h$. Because $I = A\,k^2$ (in which A is the metallic area of the cable), this explains the presence of k^2 in the expression for λ. As k becomes smaller, λ increases. Then, by Eq. 5, as λ increases, K decreases and, by Eq. 6, as K decreases, secondary stress increases. Therefore, as stated previously, as stiffness decreases, unit bending stress increases.

After at first falling into the "trap" of assuming that there was some fallacy in the author's reasoning and thereby following a dead-end trail, the writers returned to take a closer look at the reasoning, and at last found the solution to the apparent paradox:

For given lateral displacements of certain points on a beam with respect to others, unit stress does increase with stiffness. However, if we are given re-

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[a] July 1960, by Thomas Alan Wyatt (Proc. Paper 2542).

[2] Senior Design Engr., Bridge Div., John A. Roebling's Sons Div., The Colo. Fuel and Iron Corp.

[3] Chf. Engr., Bridge Div., John A. Roebling's Sons Div., The Colo. Fuel and Iron Corp.

lative rotations of certain cross sections of a beam with respect to others, unit stress is independent of stiffness. The rotations simply force changes in length of the fibers. Normally, therefore, in such a case, a change in k would not affect the secondary stress. In this particular case, however, the rotation is itself a function of the stiffness. As stiffness decreases, the relative rotation, and thereby, the secondary stress, increases. Placing this thought in the practical context of the subject problem, we see the perfectly logical inter-relation: let us consider the end panel of the cable adjacent to a tower saddle. If the cable is relatively flexible, the rotation caused at the tower saddle by changes in load is absorbed to a large extent in the first panel, and the forces tending to rotate the cable band at the end of the first panel are relatively small. Thus, the cable band rotates only a small amount and the relative rotation of the ends of the first panel of cable is almost equal to the entire rotation of the cable cross-section at the saddle.

On the other hand, if the cable is relatively stiff, the cable will force the first cable band to rotate, thereby transmitting some of the rotation into the next panel.

In considering the models presented in Figs. 1 through 3, it has occurred to the writers that another method of presentation might have made the analogy clearer to many readers.

The author makes the simplifying assumptions that, before end rotation is applied, the cable, at any panel point is straight through the cable band, that the suspender tension does not exist, and that cable tension does not change at the band. He then proceeds to express the moment existing at any panel point after end rotation as $T\, l_c\, \theta_p$.

Another way to illustrate this phenomenon is to consider the cable polygon as it is, the cable angle and tension changing at each panel point, in the presence of the suspender tension. The common assumption that the cable represents a string polygon implies that the three forces at each point, two cable tensions and a suspender tension, meet at a point located in the line of the suspender force. One could then reason that this string polygon is disturbed by the secondary influences which the author is studying. This disturbance has the affect of moving the point of intersection of the cable tension slightly toward the tower or toward mid-span a distance e (for eccentricity). The vertical reaction produced by the cable tensions at this new point of intersection would still be equal to the suspender pull (say F). The moment would then be expressed as F e.

The writers do not doubt that the author's approach lends itself better to mathematical manipulation, but the foregoing may help to clarify the physical concept.

Experience on this subject could probably be summarized by saying that no conditions have been observed that have given designers cause for concern. At the same time, many designers recognize the existence of secondary stresses of this type and consciously attempt to minimize rotation. One reason for the lack of observable affect may be the capacity of cold-drawn hot galvanized bridge wire to absorb great over-stress with little permanent elongation and with a resultant improvement in elastic properties.

A few examples from practice may be of interest:

Persons concerned with the design and construction of the Forth Road Bridge in Scotland noted that one of the saddles would undergo relatively large rotation between free-hanging cable and dead-load conditions, if conventional proce-

dures were followed. To minimize this rotation, means were found to accomplish the necessary motion partly by rotation and partly by translation.

In canvassing experience prior to reaching this decision, the following points of interest were developed:

During construction of the San Francisco-Oakland Bay Bridge, one of the saddles experienced a rotation equal to that which gave rise to the discussion on the Forth Road Bridge. No noticeable ill-effects have been observed.

At a similar saddle on the Mackinac Bridge, the designers purposely made the member supporting the saddle as long as possible, to minimize the rotation.

The writers know of no instance where broken wires have been found that could be attributed to the subject influences. Careful inspections have been made on many existing bridges without uncovering any problems of this type. One of the most complete of such surveys was that recently conducted on the Brooklyn Bridge, in preparation for its new life as a modern vehicular bridge. Over the years, sample of wires from various locations on the Brooklyn Bridge cables have been tested and found to have properties as good, if not better, than in their original state.

However, as indicated previously, specific design criteria are always better than rules-of-thumb or pure judgment and the author is encouraged to carry his thinking through to that point.

As a source of confirmation or correction of his reasoning, the author is referred to a report dealing with the Delaware River Bridge in which there will be found data having a direct bearing on the subject of this paper.[4] Correlation of these results with the mathematics of the paper would be of great value. The author may be gratified to note the experimental verification contained therein of the series of cable "kinks" shown at the cable bands in Figs. 2 and 3. It will also be noted that the experiments developed a friction coefficient of between 15% and 20% between the wires of a wrapped cable.

Under the heading "Slip Through Hanger Bands" the author states that a coefficient of friction of 60% has been suggested for the friction between the wires within a cable band. It may be that this value is the result of an erroneous interpretation of test results. Tests pertinent to this point are also included in the above-mentioned report.[5] Here it is seen that the force required to move a cable band along a parallel wire cable was approximately 60% of the total bolt tension. Several other tests on other projects have confirmed this value. It should be noted, however, that two friction surfaces are involved. Therefore, the actual coefficient of friction is one-half this ratio, or approximately 30%. These figures are artificially high with respect to other pure friction tests. The difference is generally attributed to the additional resistance to sliding caused by the bulging of the cable adjacent to the band. Thus, there is some reason to question the use even of the 30% value as an appropriate coefficient of friction for this study.

The same report[6] shows the force required to pull individual wires out of a sample of cable clamped by a cable band. In this case, the value was an average of 33.5%, corresponding to the 60% value stated previously. Thus, the corresponding true coefficient of friction would be approximately 17%. This leads

[4] "Final Report of the Board of Engineers to the Delaware River Bridge Joint Commission," pp. 124 and 125.

[5] Ibid., pp. 126 and 127.

[6] Ibid., p. 127.

one to wonder if the range of 15% to 20% should not be considered appropriate for the wires under the band as well as for wires under wrapping.

The writers hope that, for his closing discussion, the author will find time to use, or at least comment on, the suggestions contained herein, to correlate his reasoning with the available test results, and to conclude with some specific design criteria to guide the designer.

DYNAMIC RESPONSE OF ELASTO-PLASTIC FRAMES[a]

Discussion by John A. Blume

JOHN A. BLUME,[5] F. ASCE.—This paper is a notable addition to the still meager fund of information of the inelastic response of structures or idealized dynamic systems to actual earthquake records. With continued efforts in this direction sufficient data will some day be at hand to isolate and to assign reliable numbers to the many parameters of the problem.

It may be postulated that the two basic items of attenuation for the author's single mass elasto-plastic system are damping and work done in inealstic response. However, each of these factors has subdivisions and there is no doubt interaction between the sub-divisions and the basic items, as well as with resonant magnification. Changes in the displacement ratios (Figs. 5 to 11, inclusive) can be considered as indicating the values of the broad categories of damping and inelastic response within the limits the author has noted. The isolation of the two attenuating factors seems important for both research and inelastic energy design purposes.

Mr. Penzien does not present data for the cases of infinite yield point with damping at 0.05, 0.10, and 0.15 of critical. Such values would help to isolate the damping attenuation at any period and enable the elasto-plastic attenuation to be obtained by deducting that due to damping from the total. This can be done directly with the data provided for the zero damping condition and the results are most interesting. The author's data indicate that the effectiveness of the inelastic response varies (when $\lambda = 0$) with not only θ but with period, T. The reasons for this no doubt involve the proximity to the quasi-resonant magnification peak in the short period range of the El Centro 1940 earthquake spectrum.

A factor sometimes overlooked in this field of effort is that a designer may consciously or unknowingly change stiffness with strength or yield. He can also change the yield value without changing stiffness. If stiffness changes, so does period, assuming the mass essentially constant. Should a proposed structure initially designed so as to have a 1.2 sec fundamental period and initial yield value of 0.12 g be both strengthened and stiffened (in the later design stage) so that T became 1.0 sec and yield 0.15 g, the response attenuation may be essentially unchanged or even less. The reverse may be true, however, for lower yield values.

Multistory buildings, especially of the skeleton and skin type without walls, may well have lower-story yield values of 0.10 g and less. They probably have damping values, as distinguished from inelastic energy absorption, in the or-

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[a] July 1960, by J. Penzien (Proc. Paper 2545).

[5] Pres., John A. Blume & Assoc., Engrs., San Francisco, Calif.

der of 5% of critical, and certainly not over 10%. Further inelastic studies in this range would be most interesting.

A study of the author's figures indicates several matters and raises questions for future research including the following:

1. At any period near the peak of the earthquake spectrum, the relative attenuation values of damping and inelasticity vary greatly with the yield strength.
2. Damping value at the shorter periods is relatively more effective for greater θ-values than for low yield points.
3. The relative difference between 5%, 10%, or even 15% damping at any period for all except very low yield values is minor; that is, in general 10% damping offers a minor attenuation over that for 5% damping.
4. 5% damping provides, especially for large θ-values, or elastic response, considerable reduction over no damping for the entire period range.
5. For a given period and damping value, a yielding or inelastic condition tends to attenuate response otherwise (elastically) of greater magnitude than the yield value, except for very low yield values, below which the response increases.

These and other indications are in general agreement with other test data and proposed approaches to inelastic design.

The "optimum" yield values to which reference is made in Conclusions 2 and 3 should be approached with caution, as Mr. Penzien notes, until other earthquake records are tried. The N-S component of the same earthquake, for example, is generally more severe than the E-W component utilized in this work. Other earthquakes of variable epicentral distances, intensity, and soil conditions must also be considered. It is to be noted that for damping values other than 0, the optimum is not a true optimum but essentially a constant for θ-values greater than the 0.10 g to 0.20 g range. This indicates elastic response for this earthquake since the viscous damping assumed was sufficient to provide dynamic stability without inelastic excursions. An interesting question is how would 5% damping perform for the stronger N-S component of El Centro 1940?

The relatively small range from code design unit stresses to yield unit stresses combined with code base shears and the high accelerations of elastic earthquake spectra plainly indicate that inelastic response is a basic part of the earthquake resistance of most buildings.[3] Damping alone cannot do the job. More work in inelastic response, (elastic-plastic and other types) with many earthquake records, variable stiffnesses, and with multi-mass systems as well as single mass systems is obviously desirable. The profession also needs data from more static testing of members and frames, not just to yield but with adequate readings all the way to failure, and with reverse loadings and repeated cycles having various degrees of inelastic deflection. As we take "our heads out of the sand" to see the physical and economic need for emergency inelastic earthquake response we shall make progress, no doubt much more than the fabled ostrich.

TESTS ON A 120-FT SPAN PRESTRESSED CONCRETE BEAM[a]

Discussion by A. A. Eremin

A. A. EREMIN,[3] M. ASCE.—In this paper Messrs. Base and Rowe have shown interesting design and test measurements of a prestressed precast concrete beam, that are similar to those used in the overpass at the Chiswick Rotary Junction of the Great West Road, England.

The authors have proved that the assumptions used in the design of the bridge beams were in close agreement with the resulting test measurements. However, in the study of the physical and elastic properties of the test beam, it is interesting to note the statement made by the authors' in regard to the resulting tests measurements (under the heading, "Analysis and Discussion of Results: Test Results"): "after the first two cables had been stressed, the movement at the joints continued until five or six cables had been stressed."

Appearance of cracks at the joints and their closure, only after five or six cables in the beam were stressed, shows that the details at the joints require further improvement.

Furthermore, Messrs. Base and Rowe stated that the east joint was opening at a lead of approximately 67 kips per jack whereas at the midspan the crack actually commenced at slightly under 134 kips. This fact also indicates that the careful consideration for further improvement of design and method of casting of the joints between precast sections is needed.

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[a] September 1960, by G. D. Base and R. E. Rowe (Proc. Paper 2608).

[3] Cons. Structural Engr., Berkeley, Calif.

FREEZING AND THAWING EFFECTS ON PRESTRESSED CONCRETE[a]

Discussion by Walter E. Kunze

WALTER E. KUNZE,[12] M. ASCE.—One of the conclusions stated in this paper is that durability is improved by post-tensioning. Although there are ample data to substantiate this conclusion, the data presented in Table 3 do not necessarily lead to this conclusion. The average durability factor of the 3,000 psi plain concrete is shown as 66.7 for six samples. Omitting specimen D-2 (that is 53.7 points lower than any of the other five) the average becomes 77.2 that is higher than any of the average values for 5,000 psi concrete and thus it was more durable than the prestressed concrete of higher strength. This might be due to air entrainment in the "D" mix whereas other mixes may have been non air-entrained. The paper indicates that an air-entraining admixture was used but from Table 1 only mix IID appears to substantiate this. From this standpoint, the authors seem to emphasize the need for air entrainment in producing durable concrete whether or not it is prestressed.

The paper clearly shows that at a stress level of 2,000 psi compression, prestressed concrete is more durable than unstressed concrete. Prestressed concrete is commonly used in flexural members. Although the maximum prestress in a member is often 2,000 psi or higher, the average is far less. Furthermore, under design load, much of the concrete is stressed to a very low level and some is in tension. This again points out the need for durable concrete, whether or not the concrete is prestressed. For most concretes used in prestressing, air content of $5\% \pm 1\%$ is required to assure a high degree of resistance to freezing and thawing, along with low water-cement ratio and adequate curing.

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[a] October 1960, by M. J. Gutzwiller and F. E. Musleh (Proc. Paper 2633).
[12] Mgr., Structural and Railways Bur. Portland Cement Assoc.

PRINCIPLE OF VIRTUAL WORK IN STRUCTURAL ANALYSIS[a]

Discussion by Ray W. Clough

RAY W. CLOUGH,[19] F. ASCE.—There has been presented an interesting paper demonstrating the general applicability of the principle of virtual work to problems in structural mechanics. However, his presentation does not point out the dual nature of the principle, that, in the opinion of the writer, is the key to its most effective utilization. The examples presented by Mr. DiMaggio represent both aspects of the duality, but the distinction between the two forms in which the principle may be applied is not made clear. Although the dual nature of the virtual-work principle has been described before,[20] a brief restatement may be of interest.

The basic principle of virtual work has been expressed mathematically in Eq. 8. In words this principle may be stated as follows:

> "If any structure is in equilibrium under the action of a set of external forces and is subjected to a displacement which is compatible with the constraints of the structure, the work done by the external forces during this displacement is equal to the work done by the internal forces."

The word "virtual" in the name of this work principle denotes the fact that the mechanism causing the displacements is not associated with the forces acting on the structure.

The validity of the principle may be demonstrated, as Mr. DiMaggio has shown, by starting with the equations of equilibrium. Conversely, the work principle could be accepted as fundamental and the equations of equilibrium derived therefrom. Obviously, it makes little difference which point of view is adopted. The important point to note is that inherent in the application of the principle of virtual work are two requirements: The system must be in equilibrium and the displacements must be compatible with a prescribed condition of contraint. The forces and displacements need not be (and normally are not) related effects, however.

The dual nature of the virtual-work principle results from the two ways in which it may be applied. If the system is subjected to a set of forces in equilibrium, the expression of the equality of internal and external virtual work will impose the requirement of compatibility. On the other hand, if the system is subjected to a compatible deformation pattern, the equality of internal and external virtual work will express the requirements of equilibrium. Following these two procedures, the principle of virtual work may be used to evaluate either displacements of forces in a structural system.

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[a] November 1960, by Frank Di Maggio (Proc. Paper 2643).

[19] Prof. of Civ. Engrg., Univ. of California, Berkeley, Calif.

[20] "Energy Theorems and Structural Analysis," by J. H. Argyris, Aircraft Engineering, Vol. XXVI, October, 1954.

Considering the first use, the basic problem is the evaluation of certain desired displacement quantities resulting from a given condition of deformation that exists in the structure. To determine a displacement component that is compatible with these deformations, an artificial (or virtual) force system is applied. The only requirement imposed on these virtual forces is that they must be in equilibrium; they need bear no relation, whatsoever, to any actual system of forces or reactions acting in the structure. Making use of this approach, the force quantities in Eq. 8 (W, V, M) would represent a virtual force system in equilibrium, while $\eta(x)$ would be the actual deformations and compatible displacements. The mechanism causing these deformations is unimportant. As Mr. DiMaggio has pointed out, they may be elastic deformations due to actual loads, they might be visco-elastic effects, they might be caused by temperature changes, and so forth. However, in this use of the principle, it should be clearly understood that these deformations are definite, actual effects, and they should not be termed virtual. The force system is the virtual system in this case.

In the second use of the virtual force principle, the objective is the evaluation of certain specified forces (usually external) that are in equilibrium with a given (usually internal) force pattern. To determine the desired force components, an appropriate artificial (or virtual) system of displacements is introduced. The requirement imposed on these virtual deformations is that they must be compatible with the essential constraints of the system. However, they need not satisfy all of the actual support or internal continuity conditions. In fact, a displacement must be introduced deliberately at any point where a force is to be determined; and if the desired quantity is an internal force or a reaction, the displacement will violate an actual constraint (as in the last example). Making use of this approach, the deformations $\eta(x)$ in Eq. 8 would represent a virtual displacement, while the forces (W, V, M) would be an actual equilibrium force system. These forces might be due to any cause: loading, thermal changes, displacements of supports, and so forth; but they must be actual forces of known amplitude.

In his examples, Mr. DiMaggio has illustrated the two basic uses of the virtual work principle. He also has correctly demonstrated the freedom of choice that is inherent in the selection of virtual force or displacement patterns, that is, the fact that the virtual force system is independent of the actual forces and reactions in the structure, and that the virtual displacement system is not controlled by continuity requirements in the actual system. However, the specific requirements of the principle of virtual work which are summarized previously have not been clearly stated by Mr. DiMaggio.

Furthermore, the statement under the heading "Restricted Deviation for a Beam":

> "The function $\eta(x)$ does not necessarily satisfy any of the constraints on deflection of the structure acted on by W."

is true only when the principle of virtual displacements is applied, but is not true of the principle of virtual forces. For example, in any of Mr. DiMaggio's examples of deflection computations (that are applications of the principle of virtual forces), it is essential that $\eta(x)$ be the actual displacements of the structure, satisfying all boundary conditions and constraints in the portions of the structure in which deflections are to be computed.

One other word of caution is offered. In Eq. 57, the curvature term due to shear forces $\left(y_s'' = \frac{k\overline{W}}{AG}\right)$ is valid only for prismatic members; otherwise, the variations of k and A with x would lead to additional terms in the expression. In the general case of a straight beam with arbitrarily varying section, the correct expression for deflection in a flexural element (including the effect of shear deformation) is given by Eq. 65 but not by Eq. 59.

PERIODS OF FRAMED BUILDINGS FOR EARTHQUAKE ANALYSIS[a]

Discussion by Merit P. White

MERIT P. WHITE,[15] F. ASCE.—This interesting paper is useful in that it shows how easily one can take account of various kinds of flexibility (shearing and bending of the structure, and rotation and sidesway of its foundation) in computing the lower natural periods of vibration of a building. However, two limitations on its practical usefulness may be mentioned: (1) the structures that have the characteristics assumed in the derivation of Eqs. 28a and 29 are rare, and (2) the parameters that appear in these equations are easier to define than to evaluate. It may be added that to the writer these equations seem rather complicated to use.

The structure that is envisaged by the authors has the following characteristics:

(a) The weight per unit height or per story is constant throughout the structure except at the roof that is heavier than the floors. Thus, the structure is apparently considered to be of uniform section.

(b) The shear stiffness at any level is directly proportional to the total weight above that level, except in the first story that has just half the stiffness of the second story; this is unrealistic. Because the contribution of the structural frame to the stiffness of a normal building is very small compared to that of its walls and partitions it is unlikely that the shear stiffness of a structure of uniform section will increase by a factor of order between 10 and infinity in descending from the roof to the second story as it would have to do to remain proportional to the total weight above, and as is implied by the values assigned to r_s in Table 1. In a building of constant section it is more realistic to assume constant shear stiffness.

(c) The flexural stiffness increases by an amount proportional to distance below the roof. The validity of this assumption depends on the extent to which walls and partitions do not participate in resisting bending.

(d) The soil supporting the structure allows both rocking and lateral displacement proportional to base moment and base shear, respectively. Finally, the natural period of vibration corresponding to each of these types of distortion is determined and the period of vibration due to all of them combined is found by squaring, adding, and taking the square root of the sum.

The authors verify their results by computing the lower mode periods of the Alexander Building in San Francisco utilizing the structural parameters (weight, stiffnesses, and so forth) published by Blume.[13] Because the computed periods agree fairly well with those observed the authors conclude that their relations are reliable and can be used for accurately predicting periods

[a] December 1960, by Mario G. Salvadori and Ewald Heer (Proc. Paper 2680).
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of structures. This is true so far as the relation between structural parameters and periods is concerned. However, the accuracy of the result is limited by the accuracy with which the parameters are known. The stiffness parameters quoted by Blume[13] were obtained not by measurement or by direct computation but indirectly, by trial and error computations of periods and mode shapes that were repeated with different parameters until the predictions agreed with observations. If it is necessary to find periods from an examination of structural plans, the accuracy will be worse, although probably better than that of Eq. 2.

So far as the actual computation of periods from a given set of parameters is concerned the method given herein is probably one of the simplest and at the same time is accurate enough. There are no restrictions on mass and stiffness distributions.

First, a reasonable deflection curve is chosen, say a concave inward parabola if shear is the principal source of flexibility, a concave outward parabola if bending is of principal importance, or a straight line if both are about equally significant or if base rotation governs. As appears subsequently, it does not make very much difference which one is selected. Next, the distribution of maximum accelerations is tabulated or plotted; it is simply the displacement at each point multiplied by p^2-the square of the as yet unknown angular frequency p. The maximum acceleration at any section multiplied by the corresponding mass per unit height of structure gives the inertia loading per unit height. From the inertia loading the shear (V) and the bending moment (M) along the length of the building are found by graphical integration from the roof down. The elastic energy, equal to $\frac{V}{2k} + \frac{M}{2EI}$, is found as a function of height, where k is the shear stiffness as defined by the authors and E I the bending stiffness. The total elastic energy of the structure is the area beneath the elastic energy distribution curve. To this must be added the elastic energy stored in the foundation that equals $M_b^2/(2B)$ if lateral displacement is neglected, as is usually permissible. The elastic energy of the structure-foundation system will contain the unknown common factor p^4.

The maximum kinetic energy is next found. The assumed displacement curve multiplied by p gives the velocity distribution. This squared and multiplied by m/2 gives the kinetic energy distribution along the building (m is mass per unit height and need not be uniform). The area beneath the kinetic energy distribution curve is the total kinetic energy of the structure. This will contain the unknown p^2. When this system vibrates in one of its natural modes there is cyclical transformation of energy from kinetic to elastic to kinetic, and so forth. So that the maximum elastic energy is equal to the maximum kinetic energy. Equating the two energies just computed gives an expression containing the single unknown p^2, that is determined thereby.

This method becomes much simpler and is still reasonably accurate for many structures if the mass and the stiffnesses are all assumed to be constant and equal to the average values of the actual quantities. The following symbols are introduced in addition to or modifying those defined by the authors:

D = maximum displacement at the roof (in feet)

f = inertia force per unit height (in kips per foot)

I = average cross-sectional moment of inertia for flexure (in ft^4)

k = average shearing stiffness coefficient (in kip-feet per foot)

M = bending moment at any section (foot-kips)

v = maximum velocity at any section (feet per second)

Assuming the extreme displacement in the fundamental mode to be triangular, zero at ground and D at roof level, one gets the equation of motion of any element

$$y = \left(D \frac{z}{L}\right) \sin p\,t \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

Then the maximum inertia loading is

$$f = m\,p^2\,D \frac{z}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

and the maximum velocity is

$$v = p\,D \frac{z}{L} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

The shear and bending moment at any level z due to the inertia loading are

$$V = \int_z^L f\,dz_1 = \left(L^2 - z^2\right) m\,p^2 \frac{D}{2\,L} \quad \ldots\ldots\ldots\ldots \quad (33)$$

and

$$M = \int_z^L f\left(z_1 - z\right) dz_1 = \left(2\,L^3 - 3\,z\,L^2 + z^3\right) m\,p^2 \frac{D}{6\,L} \ldots\ldots \quad (34)$$

The moment at the foundation is found by letting z = 0 in Eq. 34,

$$M_b = m\,p^2\,D \frac{L^2}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

If the lateral displacement of the foundation is neglected, the total elastic energy of the system is

$$PE = \frac{1}{2\,E\,I} \int_0^L M^2\,dz + \frac{M_b^2}{2\,B} + \frac{1}{2\,k} \int_0^L V^2\,dz$$

$$= m^2\,p^4\,D^2\,L^3 \left\{0.0131 \frac{L^2}{E\,I} + 0.0555 \frac{L}{B} + \frac{0.0667}{k}\right\} \quad \ldots\ldots \quad (36)$$

The total kinetic energy is

$$KE = m\,p^2\,D^2 \frac{L}{6} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (37)$$

Equating the two energies and solving for $T^2 = 4\,\pi^2/p^2$ gives

$$T^2 = 4\,\pi^2\,m\,L^2 \left\{0.0785 \frac{L^2}{E\,I} + 0.333 \frac{L}{B} + \frac{0.40}{k}\right\} \quad \ldots\ldots \quad (38)$$

This corresponds to Eqs. 28a and 29.

For a deflection assumed to be a concave outward parabola the expression for the period becomes:

$$T^2 = 4\,\pi^2\,m\,L^2 \left\{\frac{0.0802\,L^2}{E\,I} + \frac{0.313\,L}{B} + \frac{0.357}{k}\right\} \ldots\ldots\ldots \quad (39)$$

For a quarter sine curve, convex outward, one obtains

$$T^2 = 4\pi^2 m L^2 \left\{ \frac{0.0752 L^2}{E I} + \frac{0.329 L}{B} + \frac{0.406}{k} \right\} \ldots\ldots\ldots (40)$$

When these expressions are applied to the north-south fundamental mode of the Alexander Building, using the following parameters: m = 2.2, L = 196, E = 4.32×10^6, I = 3150, B = 2.84×10^9 and k = 1.57×10^6, there result, respectively, T = 1.29 sec, T = 1.26 sec, and T = 1.28 sec.

The observed periods for this mode ranged from 1.23 to 1.27 sec.

The writer considers this method of period determination to have several advantages over the one described by the authors: (1) the assumption of constant stiffnesses is more realistic for the great majority of uniform buildings than is the assumption that shearing stiffness is proportional to the total mass above any section, (2) the expressions that have been obtained for the period of a uniform structure are simpler and easier to use than is one that requires reference to tables for values of coefficients, (3) the method can be easily generalized and applied to structures that are so nonuniform as to require special treatment.

ERRATA

Journal of the Structural Division

Proceedings of the American Society of Civil Engineers

"Load Distribution in Composite Girder-Slab Systems" by G. Hondros and J. G. Marsh, November 1960, Proc. Paper 2645.

p. 80. The caption for Fig. 2 should be as follows:

FIG. 2.—MODEL AFTER CENTRAL LONGITUDINAL CUT

p. 81. The caption for Fig. 3 should be as follows:

FIG. 3.—SLAB FORMWORK AND REINFORCEMENT

p. 102. In Eq. 7 the terms in the first set of parentheses should be as follows: $1.07 + 0.454\ E_C$. In Table 3 change 3 to 11 and 12 and 13 to 5 to 13 and 14 to 15, respectively, in line 1.

"Movements of a Cable Due to Changes in Loading" by James Michalos and Charles Bornstiel, December 1960, Proc. Paper 2674.

pp. 34 and 35. Table 5 should have its two halfs interchanged so that the half headed TABLE 5 appears on page 34 and the half headed TABLE 5.—CONTINUED appears on page 35.

PROCEEDINGS PAPERS

The technical papers published in the past year are identified by number below. Technical-division sponsorship is indicated by an abbreviation at the end of each Paper Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Pipeline (PL), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways and Harbors (WW), divisions. Papers sponsored by the Department of Conditions of Practice are identified by the symbols (PP). For titles and order coupons, refer to the appropriate issue of "Civil Engineering." Beginning with Volume 82 (January 1956) papers were published in Journals of the various Technical Divisions. To locate papers in the Journals, the symbols after the paper number are followed by a numeral designating the issue of a particular Journal in which the paper appeared. For example, Paper 2703 is identified as 2703(ST1) which indicates that the paper is contained in the first issue of the Journal of the Structural Division during 1961.

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c. Discussion of several papers, grouped by divisions.

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